

# Ludics Characterization of Multiplicative-Additive Linear Behaviours

Christophe Fouqueré \*

Myriam Quatrini †

## Abstract

Ludics is a logical theory that J.-Y. Girard developed around 2000. At first glance, it may be considered as a Brouwer-Heyting-Kolmogorov interpretation of Logic as a formula is denoted by the set of its proofs. More primitively, Ludics is a theory of interaction that models (a variant of) second-order multiplicative-additive Linear Logic. A formula is denoted by a set of objects called a behaviour, a proof by an object that satisfies some criteria. Our aim is to analyze the structure of behaviours in order to better understand and refine the usual notion of formulas or types. More precisely, we study properties that guarantee a behaviour to be recursively decomposable by means of multiplicative-additive linear connectives and linear constants.<sup>1</sup>

## 1 Introduction

Ludics is a logical theory that J.-Y. Girard developed around 2000. At first glance, it may be considered as a Brouwer-Heyting-Kolmogorov interpretation of Logic as a formula is denoted by the set of its proofs. More primitively, Ludics is a theory of interaction, where interaction is the fact that a meeting happens between two objects together with the dynamical process that this meeting creates. This notion is primitive in the sense that the main objects of Ludics, called designs, are defined with respect to interaction: they are objects between which meetings may happen and on which rewriting processes may be described. Hence, among the computational theoretical and methodological frameworks, Ludics is ontologically closer to Game Semantics than to Proof Theory or Type Theory. Indeed, if interaction corresponds to cut and cut-elimination in Proof Theory, and to application rule and normalization in Type Theory, it presupposes more primitive notions fixed: formulas and formal proofs in Proof Theory, types and terms in Type Theory. On the opposite, the concept of play in Game Theory serves as a means for interaction. However we should notice that, in Game Semantics, the definition of a game is external to interaction, whereas in Ludics the corresponding notion of behaviour is internal: it is a closure of a set of designs with respect to interaction. In other words, Ludics may be considered as an untyped computational theoretical framework, but with types subsequently recovered. In [12], Terui showed that such a notion of *interactive type* may be applied with success to the study of formal grammars. Our aim is to analyze the structure of interactive types in order to better understand and refine the usual

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\*Université Paris 13, Sorbonne Paris Cité, LIPN, F-93430, Villetaneuse, France  
CNRS, UMR 7030, F-93430, Villetaneuse, France. [christophe.fouquere@lipn.univ-paris13.fr](mailto:christophe.fouquere@lipn.univ-paris13.fr)

†IML –FRE 35-29, Université Aix-Marseille & CNRS, F-13284 Marseille Cedex 07, France.  
[myriam.quatrini@univ-amu.fr](mailto:myriam.quatrini@univ-amu.fr)

<sup>1</sup>This work has been partially funded by the French ANR projet blanc “Locativity and Geometry of Interaction” LOGOI ANR-10-BLAN-0213 02.

notion of formulas or types. More precisely, we characterize in this paper behaviours that correspond to (linear) logical formulas.

We give properties that guarantee a behaviour to be recursively decomposable by means of multiplicative-additive linear connectives and linear constants. First of all, *essential finiteness* (or *uniform boundedness* when infinite sums are accepted) ensures that such a decomposition indeed terminates on constants. Additive decomposition is immediate as it is already present in Ludics. Multiplicative decomposition is a more complex problem to tackle. For that purpose, two notions turn out to be fundamental: (1) *incarnation*, a specific concept of Ludics, enables to characterize which part of a behaviour is used during interaction, (2) the presentation of a design as a set of *paths* instead of a set of chronicles as originally introduced by Girard. These notions help also study the relation between Ludics and Game semantics [7, 8, 1]. In a previous work [9], the presentation of a design as a set of paths was the key point that allows the authors to compute the incarnation of a set of designs without computing the behaviour. To be able to fully characterize behaviours that are linearly decomposable, we investigate two new notions: *visitability* and *regularity*. A visitable path is a path that may be travelled by interaction. A regular behaviour is such that its set of visitable paths is exactly the set of positive-ended chronicles of its incarnation, stable by shuffle and dual operations, where a shuffle of two paths is an interleaving of actions that respects polarity. With that given, we prove the following result:

$\mathbf{A} \in \mathcal{C}_\infty$  iff  $\mathbf{A}$  is a regular uniformly bounded behaviour.

where  $\mathcal{C}_\infty = \mathcal{C}_\infty^+ \cup \mathcal{C}_\infty^-$ , defined inductively in the following way:

$$\mathcal{C}_\infty^+ = \mathbf{0} \mid \mathbf{1} \mid \bigoplus_{n \in [1, \infty[} (\bigotimes_{q \in [1, \infty[} \downarrow \mathcal{C}_\infty^-) \quad \mathcal{C}_\infty^- = \mathbf{\top} \mid \mathbf{\perp} \mid \bigotimes_{n \in [1, \infty[} (\wp_{q \in [1, \infty[} \uparrow \mathcal{C}_\infty^+)$$

The paper is organized as follows. In section 2, we recall the main facts concerning Ludics. In particular, we make explicit the equivalence between the two presentations of a design, as set of paths versus set of chronicles. We define what is a shuffle of paths. To our knowledge, the shuffle operation, largely used in combinatorics and to study parallelism (see for example [6, 11]), appears in Logics only to study non-commutativity [2, 3]. We give a few properties concerning orthogonality in terms of path travelling, introducing visitable paths, *i.e.*, paths that are visited by orthogonality. In section 3 additive and multiplicative operations on behaviours are studied with respect to visitability. The main result is a characterization of a tensor of behaviours mainly as a shuffle operation on paths. Section 4 is devoted to prove our main theorem already stated above.

## 2 Ludics from Paths, Incarnation from Visitable Paths

### 2.1 Chronicles and Paths

The main objects of Ludics, the **designs**, are defined by Girard [10] in order to be the support of interaction and of its dynamics. The interaction between two designs occurs when their respective **bases** share a same address in dual positions. The dynamic part of interaction is decomposed in elementary steps called **actions** (moves in Game Semantics). The (dynamics of the) interaction consists in following two dual alternate sequences of actions, one in each design.

**Definition 2.1 (Base, Action, Sequence)**

- A **base**  $\beta$  is a non-empty finite set of sequents of pairwise disjoint addresses:  $\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n$  such each  $\Delta_j$  is a finite set of addresses, at most one  $\Gamma_i$  may be

empty and the other  $\Gamma_i$  contain each exactly one address. An address is noted as a sequence of integers.

- An action  $\kappa$  is
  - either a triple: a polarity that may be positive (+) or negative (−); an address  $\xi$  that is the focus of  $\kappa$ ; a finite set of integers  $\{k_1, \dots, k_n\}$  called a ramification. When used in an interaction, the action  $\kappa$  creates the finite set of new addresses  $\{\xi.k_1, \dots, \xi.k_n\}$  on which the interaction may continue. An action with focus  $\xi.k_i$  is said justified by  $\kappa$ .
  - or the positive action daimon denoted by  $\star$ .
- A sequence of actions  $s$  is **based on**  $\beta$  if each action of  $s$  is either justified by a previous action in  $s$ , or has its focus in  $\beta$ , or is a daimon which should be, in this case, the last action of  $s$ . An action  $\kappa$  of  $s$  is **initial** when the focus of  $\kappa$  is in  $\beta$ .

For ease of reading, one may put as superscript of an action its polarity:  $\kappa^+$  is a positive action whereas  $\kappa^-$  is a negative action.

Two kinds of sequences of actions, paths and chronicles, may equivalently be used to define designs. Roughly speaking, a path is an alternate sequence that allows to recover the justification relation between actions, a chronicle is a path with an additional constraints on the justification relation: a negative action should be justified by the immediate previous action in the sequence.

## Definition 2.2 (Path, Chronicle)

- A **path**  $p$  based on  $\beta$  is a finite sequence of actions based on  $\beta$  such that
  - Alternation: The polarity of actions alternate between positive and negative.
  - Justification: A proper action is either justified, i.e., its focus is built by one of the previous actions in the sequence, or it is called **initial** with a focus in one of  $\Gamma_i$  (resp.  $\Delta_i$ ) if the action is negative (resp. positive).
  - Negative jump: (There is no jump on positive action) Let  $q\kappa$  be a subsequence of  $p$ ,
    - If  $\kappa$  is a positive proper action justified by a negative action  $\kappa'$  then there is a sub-sequence  $\alpha_0^+ \alpha_0^- \dots \alpha_n^+ \alpha_n^-$  of  $p$  such that  $\alpha_0^+ = \kappa$ ,  $\alpha_n^- = \kappa'$  and for all  $i \in \{0, \dots, n\}$ ,  $\alpha_i^+$  is immediately preceded by  $\alpha_i^-$  and  $\alpha_i^-$  is justified by  $\alpha_{i+1}^+$  in  $p$ .
    - If  $\kappa$  is an initial positive proper action then its focus belongs to one  $\Delta_i$  and either  $\kappa$  is the first action of  $p$  and  $\Gamma_i$  is empty, or  $\kappa$  is immediately preceded in  $p$  by a negative action with a focus hereditarily justified by an element of  $\Gamma_i \cup \Delta_i$ .
  - Linearity: Actions have distinct focuses.
  - Daimon: If present, a daimon ends the path. If it is the first action in  $p$  then one of  $\Gamma_i$  is empty.
  - Totality: If there exists an empty  $\Gamma_i$ , then  $p$  is non-empty and begins either with  $\star$  or with a positive action with a focus in  $\Delta_i$ .
- A **chronicle**  $c$  based on  $\beta$  is a non-empty path such that each non-initial negative action is justified by the immediate previous (positive) action.

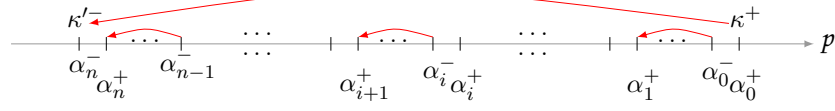


Figure 1: Path  $p$ : Constraint of *negative jump* between  $\kappa^+$  justified by  $\kappa'^-$

The polarity of a path is given by the polarity of its first action if the path is not empty: a negative path is a path with a first action that is negative. A positive-ended path is a path whose last action is positive. Abusively, we may say that an empty path is also negative or positive, a positive-ended path may be empty.

Note that the negative jump condition (see Fig.1) in the definition of a path is satisfied as soon as each non-initial negative action is justified by the immediate previous action, as it is the case with in the definition of a chronicle. Furthermore, the sequence of actions induced in the negative jump condition defines a chronicle. Such a sequence is a **view** of the path. In the other direction, a set of chronicles may give rise to a path by **shuffling** the chronicles.

**Definition 2.3 (View)** Let  $s$  be a sequence of actions based on  $\beta$ , the **view**  $\lceil s \rceil$  is the subsequence of  $s$  defined as follows:

- $\lceil \epsilon \rceil = \epsilon$  where  $\epsilon$  is the empty sequence;
- $\lceil \kappa \rceil = \kappa$ ;
- $\lceil w\kappa^+ \rceil = \lceil w \rceil \kappa^+$ ;
- $\lceil w\kappa^- \rceil = \lceil w_0 \rceil \kappa^-$  where  $w_0$  either is empty if  $\kappa^-$  is initial or is the prefix of  $w$  ending with the positive action which justifies  $\kappa^-$ .

A path  $p$  being given, we remark that  $\lceil p \rceil$  is a chronicle. We may also notice that the negative jump condition on a positive action  $\kappa^+$  justified by a negative one  $\kappa^-$  may be rephrased as follows: let  $q\kappa^+$  be the prefix of  $p$  ending on  $\kappa^+$ ,  $\kappa^- \in \lceil q \rceil$ .

If we consider the set  $p^*$  of all prefixes  $q$  of a path  $p$ , we obtain the set of chronicles  $\lceil q \rceil$  induced by  $p$ , this set is denoted  $\lceil p^* \rceil$ . Conversely, it is possible to rebuild the path  $p$  from the set of chronicles  $\lceil p^* \rceil$ . The relevant operation to build paths from chronicles is the operation of shuffle. The shuffle operation may more generally be defined on paths. The standard shuffle operation consists in interleaving sequences keeping each element and respecting order. We depart from this definition first by imposing that alternate polarities should also be satisfied, second by taking care of the daimon that should only appear at the end of a path.

**Definition 2.4 (Shuffle of paths)**

- Let  $p$  and  $q$  be two positive-ended negative paths on disjoint bases:  $\beta$  and  $\gamma$ , and such that at least one path does not end on a daimon. The **shuffle** of  $p$  and  $q$ , noted  $p \sqcup q$ , is the set of sequences  $p_1 q_1 \dots p_n q_n$ , based on  $\beta \cup \gamma$  such that:
  - each sequence  $p_i$  and  $q_i$  is either empty or a positive-ended negative path,
  - $p_1 \dots p_n = p$  and  $q_1 \dots q_n = q$ ,
  - if  $p_n$  ends with  $\star$  then  $q_n$  is empty.
- The definition is extended to paths  $p\kappa_1\star$  and  $q\kappa_2\star$  where  $p$  and  $q$  are two positive-ended negative paths on disjoint bases:

$$p\kappa_1\star \sqcup q\kappa_2\star = (p\kappa_1\star \sqcup q) \cup (p \sqcup q\kappa_2\star)$$

- The definition is extended to paths  $rp$  and  $rq$  where  $r$  is a positive-ended path and  $p$  and  $q$  are two positive-ended negative paths on disjoint bases:

$$rp \sqcup rq = r(p \sqcup q)$$

**Remark 2.5** Note that each path in  $p \sqcup q$  respects the order of actions in paths  $p$  and  $q$ . Furthermore, if at least one of  $p$  or  $q$  does not end with a daimon, each path in  $p \sqcup q$  contains exactly once all actions from  $p$  and  $q$ .

**Proposition 2.6** Let  $p$  and  $q$  be two paths such that  $p \sqcup q$  is defined, i.e., up to their (positive-ended or empty) prefix, they have disjoint bases. Let  $r$  be a sequence belonging to  $p \sqcup q$ , then  $r$  is a path.

*Proof*

- Suppose that  $p$  and  $q$  have an empty prefix and at least one of them does not end with a daimon. We check below that  $r$  satisfies criteria for being a path:
  - By construction, the conditions of alternation, justification, linearity, daimon and totality are satisfied.
  - Since all subsequences  $p_j$  and  $q_j$  start with a negative action, for all action  $\kappa$  belonging to  $r$  and such that  $w\kappa$  is a prefix of  $r$ ,  $\lceil w\kappa \rceil$  is either completely in  $p$  if  $\kappa$  is in  $p$  or in  $q$  if  $\kappa$  is in  $q$ . Then the negative jump condition on  $r$  is directly inherited from  $p$  and  $q$ .
- Suppose that  $p = p'\kappa_1\star$  and  $q = q'\kappa_2\star$  with empty common prefix. Remark that  $p'$  and  $q'$  are positive-ended paths with empty-common prefix. Furthermore either  $r \in p'\kappa_1\star \sqcup q'$  or  $r \in p\kappa_1\star \sqcup q'$ . Hence it follows from the previous item that  $r$  is a path.
- Suppose that  $p = sp'$  and  $q = sq'$  where  $s$  is a positive-ended path and  $p'$  and  $q'$  are two positive-ended negative paths on disjoint bases. The same reasoning as in the first item applies as the prefix  $s$  may be viewed either as in  $p$  or  $q$ .

It is possible to build paths from a given set of chronicles, provided that these chronicles are pairwise *coherent*. Indeed, coherence ensures that, after a common positive-ended prefix, chronicles are made of negative paths either on disjoint bases or with first actions of same focus.

### Definition 2.7 (Coherence)

- Two chronicles  $\mathbf{c}_1$  and  $\mathbf{c}_2$  of same base are **coherent**, noted  $\mathbf{c}_1 \supset \mathbf{c}_2$ , when the two following conditions are satisfied:
  - Comparability: Either one extends the other or they first differ on negative actions, i.e., if  $w\kappa_1 \supset w\kappa_2$  then either  $\kappa_1 = \kappa_2$  or  $\kappa_1$  and  $\kappa_2$  are negative actions.
  - Propagation: When they first differ on negative actions and these negative actions have distinct focuses then the focuses of following actions in  $\mathbf{c}_1$  and  $\mathbf{c}_2$  are pairwise distinct, i.e., if  $w(-, \xi_1, I_1)w_1\sigma_1 \supset w(-, \xi_2, I_2)w_2\sigma_2$  with  $\xi_1 \neq \xi_2$  then  $\sigma_1$  and  $\sigma_2$  have distinct focuses.
- Two paths  $p_1$  and  $p_2$  of same base are **coherent**, noted  $p_1 \supset p_2$ , if  $\lceil p_1^* \rceil \cup \lceil p_2^* \rceil$  is a clique with respect to  $\supset$ .

Let  $p$  be a path, it follows from the definition of a view and the coherence relation that  $\ulcorner p \urcorner$  as well as  $\ulcorner p^* \urcorner$  are cliques of chronicles.

### Definition 2.8

- Let  $P$  and  $Q$  be two sets of positive-ended paths, the shuffle of  $P$  and  $Q$ , noted  $P \sqcup Q$ , is the set  $\bigcup (p \sqcup q)$ , the union being taken on paths  $p \in P, q \in Q$  where  $p \sqcup q$  is defined.
- Let  $P$  be a set of positive-ended paths, the shuffle closure of  $P$ , noted  $P^\sqcup$ , is the smallest set such that  $P^\sqcup = P^\sqcup \sqcup P^\sqcup$  and  $P \subset P^\sqcup$ .

## 2.2 Designs as Sets of Chronicles or Sets of Paths

The notions of cliques of chronicles or cliques of paths are both relevant to define **designs**. The first one is closer to a tree-like presentation of formal proofs, the second one is closer to the definition of a strategy as set of plays. The first one was the one introduced by Girard in [10].

### Definition 2.9 (Design, Slice, Net)

- A **design**  $\mathfrak{D}$ , based on  $\Gamma \vdash \Delta$ , is a prefix-closed clique of chronicles based on  $\Gamma \vdash \Delta$ , such that chronicles without extension in  $\mathfrak{D}$  end with a positive action and the clique is non empty when the base is positive, i.e., when  $\Gamma$  is empty.
- A **slice** is a design  $\mathfrak{S}$  such that if  $w(-, \xi, I_1), w(-, \xi, I_2) \in \mathfrak{S}$  then  $I_1 = I_2$ .
- A **net of designs** is a finite set of designs such that foci of the bases do not appear twice on the same part of sequents.

Abusively, in the following, we consider also a net of designs as a set of chronicles, recovering the designs as the maximal cliques of this set of chronicles.

**Example 2.10** Let us consider the following design  $\mathfrak{D}$ , drawn in a proof-like manner<sup>2</sup>:

$$\begin{array}{c}
 \frac{\xi 000 \vdash \sigma 00}{\vdash \sigma 00, \xi 00} (+, \xi 00, \{0\}) \\
 \frac{\sigma 0 \vdash \xi 00}{\vdash \xi 00, \sigma} (+, \sigma, \{0\}) \quad \frac{\xi 100 \vdash}{\vdash \xi 10} (+, \xi 10, \{0\}) \\
 \frac{\vdash \xi 00, \sigma}{\xi 0 \vdash \sigma} (-, \xi 0, \{0\}) \quad \frac{\vdash \xi 10}{\xi 1 \vdash} (-, \xi 1, \{0\}) \\
 \hline
 \vdash \xi, \sigma \quad (+, \xi, \{0, 1\})
 \end{array}$$

The maximal chronicles are  $\mathfrak{c}$  and  $\mathfrak{d}$  given below, i.e.,  $\mathfrak{D} = \mathfrak{c}^* \cup \mathfrak{d}^* \setminus \{\epsilon\}$ :

$$\mathfrak{c} = (+, \xi, \{0, 1\})(-, \xi 0, \{0\})(+, \sigma, \{0\})(-, \sigma 0, \{0\})(+, \xi 00, \{0\}),$$

$$\mathfrak{d} = (+, \xi, \{0, 1\})(-, \xi 1, \{0\})(+, \xi 10, \{0\}).$$

The maximal paths are obtained as shuffles of these chronicles. Namely:

$$p = (+, \xi, \{0, 1\})(-, \xi 0, \{0\})(+, \sigma, \{0\})(-, \sigma 0, \{0\})(+, \xi 00, \{0\})(-, \xi 1, \{0\})(+, \xi 10, \{0\}).$$

$$q = (+, \xi, \{0, 1\})(-, \xi 0, \{0\})(+, \sigma, \{0\})(-, \xi 1, \{0\})(+, \xi 10, \{0\})(-, \sigma 0, \{0\})(+, \xi 00, \{0\}).$$

$$r = (+, \xi, \{0, 1\})(-, \xi 1, \{0\})(+, \xi 10, \{0\})(-, \xi 0, \{0\})(+, \sigma, \{0\})(-, \sigma 0, \{0\})(+, \xi 00, \{0\}).$$

Chronicles  $\mathfrak{c}$  and  $\mathfrak{d}$  (and their prefixes) are obtained as views of paths  $p, q, r$  (and their prefixes). In particular,  $\mathfrak{c} = \ulcorner q \urcorner = \ulcorner r \urcorner$  and  $\mathfrak{d} = \ulcorner p \urcorner$ .

We may notice on this example that shuffles of paths are paths, as we stated in proposition 2.6. For example, the sequence  $s = (+, \xi, \{0, 1\})(-, \xi 0, \{0\})(+, \sigma, \{0\})(-, \xi 1, \{0\})(+, \xi 00, \{0\})$ , which

<sup>2</sup>For ease of reading, throughout the paper, designs are drawn in a proof-like way with actions omitted.

interleaves chronicles  $\mathfrak{c}$  and  $\mathfrak{d}$ , does not keep all actions of  $\mathfrak{c}$  and  $\mathfrak{d}$ . Note also that it satisfies all constraints to be a path except the negative jump condition:  $(+, \xi 00, \{0\})$  is justified by  $(-, \xi 0, \{0\})$ , however one cannot find a sequence  $(\alpha_i)$  that satisfies the condition.

We recall below a result we proved in [9] that states that a non-empty clique of non-empty paths may give rise to a net of designs. Furthermore it follows from proposition 2.6 that  $\mathfrak{R}^\downarrow$  is a set of paths when  $\mathfrak{R}$  is a net of designs. Hence that makes explicit the link between paths and chronicles of a design, and more generally of a net of designs, hence justifies the switch from/to the reading of designs or nets as cliques of chronicles to/from the reading of designs or nets as cliques of paths. Thus we say that  $p$  is a positive-ended path of a net  $\mathfrak{R}$  whenever  $p$  is in  $\mathfrak{R}^\downarrow$ .

**Proposition 2.11** [9] *Let  $V$  be a non-empty clique of non-empty paths based on  $\beta$  such that maximal ones are positive-ended and let  $V^*$  be the closure by prefixes of  $V$ . The set of chronicles  $\ulcorner V^* \urcorner$  defined as the union of views of paths of  $V^*$  forms a net of designs based on  $\beta$ .*

**Proposition 2.12** *Let  $p$  be a positive-ended path of a net of designs  $\mathfrak{R}$  then  $p \in \mathfrak{R}^\downarrow$ .*

*Proof* Let  $p$  be a positive-ended path of  $\mathfrak{R}$ , we prove that  $p \in \mathfrak{R}^\downarrow$  by induction on the length of  $p$ .

- If  $p = \epsilon$  or  $p = \kappa^+$ , the result is obvious as in the two cases  $\ulcorner p \urcorner = \{p\}$ .
- Otherwise  $p$  is of the form  $w\kappa^-\kappa^+$ . As  $p$  is a path,  $\ulcorner w\kappa^-\kappa^+ \urcorner = w'\kappa^-\kappa^+$ , i.e.,  $\kappa^-\kappa^+$  are the two last actions of a (positive-ended) chronicle of  $\mathfrak{R}$ . Let us define a sequence  $q$  as  $p$  where one deletes actions in  $w'\kappa^-\kappa^+$ . It is straightforward to prove that  $q$  is a path of  $\mathfrak{R}$  and that  $r \in q \Downarrow w'\kappa^-\kappa^+$ . Thus the result follows by induction hypothesis.

Note that a path  $p$  made of actions present in a net of designs  $\mathfrak{R}$  may not be a path of  $\mathfrak{R}$ .

**Example 2.13** *Let us consider the following design  $\mathfrak{E}$  drawn on the left:*

$$\begin{array}{c} \hline \vdash \xi 000, \xi 111 \\ \hline \xi 11 \vdash \xi 000 \\ \hline \vdash \xi 000, \xi 1 \\ \hline \xi 00 \vdash \xi 1 \\ \hline \vdash \xi 0, \xi 1 \\ \hline \xi \vdash \end{array} \quad \spadesuit$$

*The two following sequences are paths:*

$$p = (-, \xi, \{0, 1\})(+, \xi 0, \{0\})(-, \xi 00, \{0\})(+, \xi 1, \{1\})(-, \xi 11, \{1\})\spadesuit.$$

$$q = (-, \xi, \{0, 1\})(+, \xi 1, \{1\})(-, \xi 11, \{1\})(+, \xi 0, \{0\})(-, \xi 00, \{0\})\spadesuit.$$

*However  $p \in \mathfrak{E}^\downarrow$  and  $q \notin \mathfrak{E}^\downarrow$ .*

### 2.3 Duality and Reversible Paths

We define in the next subsection the fundamental notion of interaction. Two designs may then be orthogonal as soon as their interaction behaves ‘well’ in a sense that will be made clear later. The closure by bi-orthogonality of a set of designs allows to recover the notion of type, called in Ludics *behaviour*. The study of these behaviours is in some aspects more graspable when interaction is defined on designs presented as cliques of paths. It is the case for visitability that characterizes sequences of actions visited during an interaction, or incarnation that defines designs of a behaviour that are fully visited by interaction. Indeed, as we shall prove in lemma 2.20, the sequence of actions visited during an interaction is a path. Furthermore paths visited during an interaction between a design and another design (or a net) are duals in the following sense.

**Definition 2.14 (Duality, Reversibility)** *Let  $p$  be a positive-ended alternate sequence.*

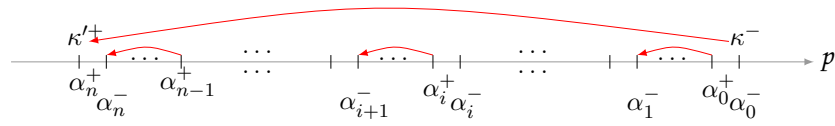


Figure 2: Path  $p$ : Constraint of *restrictive negative jump* between  $\kappa^-$  justified by  $\kappa'^+$

- The dual of  $p$  (possibly empty) is the positive-ended alternate sequence of actions  $\tilde{p}$  (possibly empty) such that<sup>3</sup>:
  - If  $p = w\star$  then  $\tilde{p} := \overline{w}$ .
  - Otherwise  $\tilde{p} := \overline{p\star}$ .
- When  $p$  and  $\tilde{p}$  are positive-ended paths, we say that  $p$  is **reversible**.

**Example 2.15** There exist paths such that their duals are not paths. Let us consider the following design:

|                            |                  |
|----------------------------|------------------|
| $\xi 000 \vdash \sigma 00$ |                  |
| $\vdash \sigma 00, \xi 00$ |                  |
| $\sigma 0 \vdash \xi 00$   | $\xi 100 \vdash$ |
| $\vdash \xi 00, \sigma$    | $\vdash \xi 10$  |
| $\xi 0 \vdash \sigma$      | $\xi 1 \vdash$   |
| $\vdash \xi, \sigma$       |                  |

The sequence  $s = (+, \xi, \{0, 1\})(-, \xi 0, \{0\})(+, \sigma, \{0\})(-, \xi 1, \{0\})(+, \xi 10, \{0\})$  is a path based on  $\vdash \xi, \sigma$ . On the contrary its dual  $\tilde{s}$  is not a path: it does not satisfy the negative jump condition.

In fact, when  $p$  is a path, the sequence  $\tilde{p}$  satisfies automatically alternation, justification, totality, and daimon conditions. The only condition which may be not satisfied is the negative jump condition. In [9] we call “restrictive negative jump condition” the fact that  $p$  is reversible, *i.e.*,  $p$  is a path such that  $\tilde{p}$  satisfies the negative jump condition (see Fig. 2). The restrictive negative jump constraint is nothing but the dual of the negative jump constraint. In the next lemma we prove that this restrictive negative jump constraint is always satisfied on chronicles.

**Lemma 2.16** *The dual of a chronicle is a path.*

*Proof* Let  $\mathbf{c}$  be a chronicle. By definition of  $\tilde{\mathbf{c}}$ , the sequence is alternate, justified and the daimon may only be the last action of the sequence. The negative jump condition results directly from the definition of chronicles. Finally, if  $\kappa^+$  is justified by  $\kappa^-$  in  $\tilde{\mathbf{c}}$  then  $\overline{\kappa^+}$  is justified by  $\overline{\kappa^-}$  in  $\mathbf{c}$ . Thus  $\overline{\kappa^-}$  immediately precedes  $\overline{\kappa^+}$  in  $\mathbf{c}$  as  $\mathbf{c}$  is a chronicle. Hence, if  $w\kappa^+$  is the prefix of  $\tilde{\mathbf{c}}$ , since  $\lceil w\kappa^+ \rceil = \lceil w \rceil \kappa^+$  and since  $\kappa^-$  is the last action of  $w$ ,  $\kappa^- \in \lceil w \rceil$ . This proves that restrictive negative jump is satisfied. Hence  $\tilde{\mathbf{c}}$  is a path.

We proved in proposition 2.6 that being a path is stable by the shuffle operation. However, as we show in next example, being a reversible path is not stable by shuffle.

**Example 2.17** *Let us consider the design drawn on the left:*

|                            |                  |
|----------------------------|------------------|
| $\xi 000 \vdash \sigma 00$ |                  |
| $\vdash \sigma 00, \xi 00$ |                  |
| $\sigma 0 \vdash \xi 00$   | $\xi 100 \vdash$ |
| $\vdash \xi 00, \sigma$    | $\vdash \xi 10$  |
| $\xi 0 \vdash \sigma$      | $\xi 1 \vdash$   |
| $\vdash \xi, \sigma$       |                  |

The path  $s = (+, \xi, \{0, 1\})(-, \xi 0, \{0\})(+, \sigma, \{0\})(-, \xi 1, \{0\})(+, \xi 10, \{0\})$  is in the shuffle of the two chronicles  $(+, \xi, \{0, 1\})(-, \xi 0, \{0\})(+, \sigma, \{0\})$  and  $(+, \xi, \{0, 1\})(-, \xi 1, \{0\})(+, \xi 10, \{0\})$ . As we prove it in lemma 2.16, the duals of these two chronicles are paths. But  $\tilde{s}$  is not a path.

<sup>3</sup>The notation  $\overline{\kappa}$  is simply  $(\pm, \xi, I) = (\mp, \xi, I)$  and may be extended on sequences by  $\overline{\epsilon} = \epsilon$  and  $\overline{w\kappa} = \overline{w}\overline{\kappa}$ .



Nevertheless, not being reversible is stable by shuffle as shown in the next lemma.

**Lemma 2.18** *Let  $p$  and  $q$  be two positive-ended paths and  $r \in p \sqcup q$  be a path such that its dual  $\tilde{r}$  is a path, then duals  $\tilde{p}$  and  $\tilde{q}$  are paths.*

*Proof* We prove the result by contradiction: let us suppose that  $p$  and  $q$  are two paths,  $r \in p \sqcup q$  is a path such that its dual  $\tilde{r}$  is a path, and at least one of the duals  $\tilde{p}$  and  $\tilde{q}$  is not a path. Without loss of generality, we can suppose that  $\tilde{p}$  is not a path.

- Remark the following. Let  $p, q, r$  defined as above then there exist  $p', q', r'$  satisfying the same requirements as for  $p, q, r$  and such that  $r' = w\star$  where  $w$  is a prefix of  $r$ . Indeed, as  $\tilde{p}$  is not a path, there exists an action  $\kappa^-$  justified by  $\kappa^+$  such that  $p = w_1\kappa^+w_2\kappa^-w_3$  and the (Restrictive Negative Jump) constraint is not satisfied on  $\kappa^-$ . Hence  $p' := w_1\kappa^+w_2\kappa^-\star$  is a path such that  $\tilde{p'}$  is not a path. Let  $r = x_1\kappa^-x_2$  and note that  $w_1\kappa^+w_2$  is a subsequence of  $x_1$ . Let  $q'$  be the subsequence of  $x_1\kappa^-$  with actions in  $q$ , then  $q'$  is a positive-ended prefix of  $q$ , hence a path. Finally let  $r' := x_1\kappa^-\star$  then  $r'$  and  $\tilde{r'}$  are paths such that  $r' \in p' \sqcup q'$  (by following the same construction steps as for  $r$ ).
- Hence for proving the lemma, it suffices to consider triples  $(p', q', r')$  satisfying the following:  $r' \in p' \sqcup q'$  is such that  $\tilde{r'}$  is a path ( $r'$  satisfies the restrictive negative jump),  $r' = w\kappa^-\star$  and the restrictive negative jump is not satisfied on action  $\kappa^-$  present in  $p'$ .
- Remark also that if lengths of  $p'$  and  $q'$  are less or equal to 2 then  $\tilde{p'}$  and  $\tilde{q'}$  are paths.
- Let  $(p_0, q_0, r_0)$  be such a triple with length of  $r_0$  minimal with respect to all such triples  $(p', q', r')$ . Notice that  $\kappa^-$  is not initial, otherwise (Restrictive Negative Jump) would be satisfied for  $p_0$ . As  $r_0$  satisfies the (Restrictive Negative Jump), there exists a sequence  $(\alpha_n^+ = \kappa^+, \alpha_n^-, \dots, \alpha_0^+, \alpha_0^- = \kappa^-)$  where  $\kappa^+$  justifies  $\kappa^-$  and the sequence is defined as in the definition of (Restrictive Negative Jump). Let us suppose  $r_0 = w'\alpha_1^-w''\alpha_0^+\kappa^-\star$ .
  - Suppose  $\alpha_0^+$  is an action of  $p_0$ , then it is also the case for its justifier  $\alpha_1^-$ . Define  $r_1 = w'\kappa^-\star$ . Remark that  $r_1$  is a path and its dual is also a path. Furthermore, we can define  $q_1$  (resp.  $p_1$ ) as the subsequence of  $q_0$  (resp.  $p_0$ ) present in  $r_1$ . Remark that  $r_1 \in p_1 \sqcup q_1$  and  $\tilde{p_1}$  is not a path. This contradicts the fact that  $r_0$  is minimal.
  - Otherwise  $\alpha_0^+$  is an action of  $q_0$ , then it is also the case for its justifier  $\alpha_1^-$ . If actions in  $w''$  are actions of  $q_0$ , we define  $r_1, q_1, p_1$  as before and this yields a contradiction. Else let  $\beta^+$  be the last action of  $p_0$  in  $w''$ . There is also an action  $\gamma^-$  of  $p_0$  which immediately precedes  $\beta^+$  in  $w''$ . One can delete from  $r_0$  the actions  $\gamma^-$  and  $\beta^+$ . Then we get a shorter sequence  $r_1$  together with paths  $p_1$  and  $q_1$  such that  $p_1$  does not satisfy the (Restrictive Negative Jump). Hence a contradiction.

## 2.4 Interaction on Designs as Clique of Chronicles or as Cliques of Paths

The interaction, also named normalization, is defined by Girard in [10] when designs are presented as cliques of chronicles:

**Definition 2.19 (Closed Cut-net, Interaction, Orthogonality, Behaviour)**

- Let  $\mathfrak{R}$  be a net of designs,  $\mathfrak{R}$  is a **closed cut-net** if
  - addresses in bases are either distinct or present twice, once in a left part of a base and once in a right part of another base,
  - the net of designs is acyclic and connex with respect to the graph of bases and cuts.

An address presents in a left part and in a right part defines a **cut**. Note that a closed cut-net has a unique design with positive base, called its **main design**.

- **Interaction on closed cut-nets** Let  $\mathfrak{R}$  be a closed cut-net. The design resulting from interaction, denoted by  $\llbracket \mathfrak{R} \rrbracket$ , is defined in the following way: let  $\mathfrak{D}$  be the main design of  $\mathfrak{R}$ , with first action  $\kappa$ ,
  - if  $\kappa$  is a daimon, then  $\llbracket \mathfrak{R} \rrbracket = \{\star\}$ ,
  - otherwise  $\kappa$  is a proper positive action  $(+, \sigma, I)$  such that  $\sigma$  is part of a cut with another design with last rule  $(-, \sigma, \mathcal{N})$  (aggregating ramifications of actions on the same focus  $\sigma$ ):
    - \* If  $I \notin \mathcal{N}$ , then interaction fails.
    - \* Otherwise, interaction follows with the connected part of subdesigns obtained from  $I$  with the rest of  $\mathfrak{R}$ .
- **Orthogonality**
  - Two designs  $\mathfrak{D}$  and  $\mathfrak{E}$  respectively based on  $\vdash \xi$  and  $\xi \vdash$  are said to be **orthogonal** when  $\llbracket \mathfrak{D}, \mathfrak{E} \rrbracket = \{\star\}$ .
  - Let  $\mathfrak{D}$  be a design of base  $\xi \vdash \sigma_1, \dots, \sigma_n$  (resp.  $\vdash \sigma_1, \dots, \sigma_n$ ), let  $\mathfrak{R}$  be the net of designs  $(\mathfrak{A}, \mathfrak{B}_1, \dots, \mathfrak{B}_n)$  (resp.  $\mathfrak{R} = (\mathfrak{B}_1, \dots, \mathfrak{B}_n)$ ), where  $\mathfrak{A}$  has base  $\vdash \xi$  and  $\mathfrak{B}_i$  has base  $\sigma_i \vdash$ ,  $\mathfrak{R}$  belongs to  $\mathfrak{D}^\perp$  if  $\llbracket \mathfrak{D}, \mathfrak{R} \rrbracket = \{\star\}$ .
  - Let  $\mathsf{E}$  be a set of designs of the same base,  $\mathsf{E}^\perp = \bigcap_{\mathfrak{D} \in \mathsf{E}} \mathfrak{D}^\perp$ .
  - A set of designs  $\mathsf{E}$  is a **behaviour** if  $\mathsf{E} = \mathsf{E}^{\perp\perp}$ .

With the following lemma, we prove that interaction gives rise to two dual paths.

**Lemma 2.20** Let  $\mathfrak{D}$  be a design of base  $\xi \vdash \sigma_1, \dots, \sigma_n$  (resp.  $\vdash \sigma_1, \dots, \sigma_n$ ), let  $\mathfrak{R}$  be the net of designs  $(\mathfrak{A}, \mathfrak{B}_1, \dots, \mathfrak{B}_n)$  (resp.  $\mathfrak{R} = (\mathfrak{B}_1, \dots, \mathfrak{B}_n)$ ), where  $\mathfrak{A}$  has base  $\vdash \xi$  and  $\mathfrak{B}_i$  has base  $\sigma_i \vdash$ ,  $\mathfrak{R}$  belongs to  $\mathfrak{D}^\perp$  iff there is a path  $p$  such that  $p$  is a path of  $\mathfrak{D}$  and  $\tilde{p}$  is a path of  $\mathfrak{R}$ . We denote the path  $p$  by  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  and  $\tilde{p}$  by  $\langle \mathfrak{R} \leftarrow \mathfrak{D} \rangle$ .

*Proof* Remark first that with  $\mathfrak{D}$  and  $\mathfrak{R}$  as given in the lemma, we have that  $(\mathfrak{D}, \mathfrak{R})$  is a closed cut-net.

- Suppose that  $(\mathfrak{D}, \mathfrak{R})$  is a convergent closed cut-net. The sequence  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$ , is built by induction on the number  $n$  of normalization steps:
  - Case  $n = 1$ : If the interaction stops in one step: either  $\mathfrak{D} = \{\star\}$ , in this case  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle = \star$ , or the main design (which is not  $\mathfrak{D}$ ) is equal to  $\{\star\}$  and in this case  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  is the empty sequence. Otherwise let  $\kappa^+$  be the first action of the main design. The first action of  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  is  $\kappa^+$  if  $\mathfrak{D}$  is the main design and is  $\overline{\kappa^+}$  otherwise.
  - Case  $n = p + 1$ : the prefix  $\kappa_1 \dots \kappa_p$  of  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  is already defined.

- Either the interaction stops, hence the main design is  $\star$ , and  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle = \kappa_1 \dots \kappa_p$  if the main design is a subdesign of  $\mathfrak{R}$ , or  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle = \kappa_1 \dots \kappa_p \star$  if the main design is a subdesign of  $\mathfrak{D}$ .
- Or, let  $\kappa^+$  be the first proper action of the closed cut-net obtained after step  $p$ ,  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  begins with  $\kappa_1 \dots \kappa_p \overline{\kappa^+}$  if the main design is a subdesign of  $\mathfrak{R}$ , or it begins with  $\kappa_1 \dots \kappa_p \kappa^+$  if the main design is a subdesign of  $\mathfrak{D}$ .

We then check by induction on its length that the sequence  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  is a path:

- The base case of the induction depends on the polarity of  $\mathfrak{D}$ : If  $\mathfrak{D}$  has a negative base then the base case is the empty sequence, and the empty sequence is a path on  $\mathfrak{D}$ , otherwise  $\mathfrak{D}$  has a positive base hence there exists a first action in the sequence  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$ , this action being the first action of  $\mathfrak{D}$ , hence a path on  $\mathfrak{D}$ .
- Suppose  $\kappa_1 \dots \kappa_p \kappa$  is a prefix of  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  and that by induction hypothesis  $\kappa_1 \dots \kappa_p$  is a path on  $\mathfrak{D}$ .
  - If  $\kappa$  is a positive action then, with respect to normalization,  $\lceil \kappa_1 \dots \kappa_p \rceil \kappa$  is a chronicle of  $\mathfrak{D}$  that extends  $\lceil \kappa_1 \dots \kappa_p \rceil$ , hence  $\kappa_1 \dots \kappa_p \kappa$  is a path on  $\mathfrak{D}$ .
  - If  $\kappa$  is an initial negative action hence  $\mathfrak{D}$  is negative and  $\kappa$  is the first action of the normalization, *i.e.*,  $p = 0$ , and  $\kappa$  is a path on  $\mathfrak{D}$ .
  - Otherwise the focus of the negative action  $\kappa$  has been created during normalization by a positive action present in  $\kappa_1 \dots \kappa_p$ , hence  $\kappa_1 \dots \kappa_p \kappa$  is a path on  $\mathfrak{D}$ .

Normalization being defined symmetrically between  $\mathfrak{D}$  and  $\mathfrak{R}$ , we also have that  $\langle \mathfrak{R} \leftarrow \mathfrak{D} \rangle$  is a path.

- Suppose that there is a path  $p$  of  $\mathfrak{D}$ , such that  $\tilde{p}$  is a path of  $\mathfrak{R}$ . Note that one of  $p$  or  $\tilde{p}$  ends with a daimon. The proof that the closed cut-net  $(\mathfrak{D}, \mathfrak{R})$  normalizes to  $\{\star\}$  is done by induction on the length  $k$  of a prefix of  $p$ . More precisely, we prove that normalization is finished when  $k = n$  if  $\mathfrak{D}$  ends with a daimon otherwise when  $k = n + 1$ . Furthermore, after  $k$  steps, interaction is done on two nets  $\mathfrak{S}$  and  $\mathfrak{T}$  such that  $p = \kappa_1 \dots \kappa_k q$ , the prefix  $\kappa_1 \dots \kappa_k$  is visited in  $\mathfrak{D}$  during these  $k$  steps and  $q$  is a path in the net of designs  $\mathfrak{S}$  on which interaction may continue:  $\mathfrak{S}$  is a subnet of  $\mathfrak{D}$ . Similarly,  $\tilde{p} = \overline{\kappa_1 \dots \kappa_k} \tilde{q}$ , the prefix  $\overline{\kappa_1 \dots \kappa_k}$  is visited in  $\mathfrak{R}$  during these  $m$  steps and  $\tilde{q}$  is a path in the net of designs  $\mathfrak{T}$  on which interaction may continue:  $\mathfrak{T}$  is a subnet of  $\mathfrak{R}$ .
  - Case  $k = n$  and  $\mathfrak{D}$  ends with a daimon. By induction hypothesis,  $\star$  is a (first) action the net  $\mathfrak{S}$  that remains of  $\mathfrak{D}$ . Thus normalization ends.
  - Case  $k = n + 1$  and  $\mathfrak{D}$  does not end with a daimon. Hence  $\tilde{q}$  ends with a daimon. By induction hypothesis,  $\star$  is a (first) action the net  $\mathfrak{T}$  that remains of  $\mathfrak{R}$ . Thus normalization ends.
  - Case  $k = 0$  with  $k \neq n$ : The induction proposition follows from the hypotheses.
  - Case  $k + 1$  with  $k + 1 \neq n$  and induction hypotheses are satisfied for  $k$ : let  $q = \kappa_{k+1} q'$ , hence  $\tilde{q} = \overline{\kappa_{k+1}} \tilde{q}'$ . Neither  $\kappa_{k+1}$  nor  $\overline{\kappa_{k+1}}$  are the daimon. Without loss of generality, we can suppose that  $\kappa_{k+1}$  is a positive action. As  $q$  is a path of the net  $\mathfrak{S}$ , the main design of the cut-net  $(\mathfrak{S}, \mathfrak{T})$  is in the net  $\mathfrak{S}$ . We have also that  $\overline{\kappa_{k+1}}$  is a first negative action of a design in the net  $\mathfrak{T}$ . We can then apply one step of normalization. The process of normalization continue with a subnet  $\mathfrak{S}'$  of  $\mathfrak{S}$  and a subnet  $\mathfrak{T}'$  of  $\mathfrak{T}$ . Remark that  $q'$  is a path of  $\mathfrak{S}'$  and  $\tilde{q}'$  is a path on  $\mathfrak{T}'$ . Hence the result.

We finish this subsection with what could be considered at first sight of low-interest. However we make use of it in the following situation: let us give a path  $p$  in a behaviour  $\mathbf{A}$ , then  $\ulcorner p \urcorner$  is a design. It follows from the proposition given below that its completion  $\ulcorner p \urcorner^c$  is a design in  $\mathbf{A}$  whereas it may not be the case for the design  $\ulcorner p \urcorner$  itself.

**Definition 2.21 (Completion of designs)** *Let  $\mathcal{D}$  be a design, the completion of  $\mathcal{D}$ , noted  $\mathcal{D}^c$ , is the design obtained from  $\mathcal{D}$  by adding chronicles  $\mathfrak{c}(-, \xi, I)\star$  whenever  $\mathfrak{c} \in \mathcal{D}$ ,  $I$  is a finite set of integers,  $\mathfrak{c}(-, \xi, I)\star$  is a chronicle and  $\mathfrak{c}(-, \xi, I) \notin \mathcal{D}$ .*

**Proposition 2.22** *Let  $\mathcal{D}$  be a design in a behaviour  $\mathbf{A}$ , consider a design  $\mathcal{C} \subset \mathcal{D}$  then  $\mathcal{C}^c \in \mathbf{A}$ .*

*Proof* Let  $\mathcal{C} \in \mathbf{A}^\perp$ . Hence  $\mathcal{C} \perp \mathcal{D}$ . Let  $p$  be the longest positive-ended path in the design  $\mathcal{C}$  that is a prefix of  $\langle \mathcal{D} \leftarrow \mathcal{C} \rangle$ . Either  $p = \langle \mathcal{D} \leftarrow \mathcal{C} \rangle$ , hence  $\mathcal{C} \perp \mathcal{C}$ , and also  $\mathcal{C} \perp \mathcal{C}^c$ . Or there exist actions  $\kappa^-$ ,  $\kappa^+$  and a sequence  $w$  such that  $\langle \mathcal{D} \leftarrow \mathcal{C} \rangle = p\kappa^-\kappa^+w$ . Consider the chronicle  $\mathfrak{c}$  such that  $\ulcorner p\kappa^- \urcorner = \mathfrak{c}\kappa^-$ . By construction,  $\mathfrak{c} \in \mathcal{C}$ . Either  $\mathfrak{c}\kappa^- \in \mathcal{C}$  hence also  $\mathfrak{c}\kappa^-\kappa^+ \in \mathcal{C}$  as  $\mathcal{C} \subset \mathcal{D}$  and there is a unique positive action after a negative action. Contradiction as  $p$  is then not maximal. Or  $\mathfrak{c}\kappa^- \star \in \mathcal{C}^c$  hence  $\mathcal{C} \perp \mathcal{C}^c$ .

## 2.5 Visitable paths, Behaviours, Incarnation

This subsection is devoted to a few properties concerning behaviours. Hence, we always suppose that a set of designs is closed with respect to bi-orthogonality. Visitability and incarnation are important concepts to study. A visitable path in a behaviour, *i.e.*, in a design of this behaviour, is a path that may be travelled by interaction. It follows from lemma 2.20 that a path is visitable iff its dual is visitable. The incarnation of a behaviour may be characterized as the subset of its designs that is fully travelled by interaction. It follows it is sufficient to study visitability and incarnation to be able to characterize behaviours and operations on behaviours. As we already mentionned in the previous subsection, all these properties rely mainly on the concept of path.

**Definition 2.23 (Visitability)** *Let  $\mathbf{A}$  be a behaviour.*

- *Let  $p$  be a path of  $\mathbf{A}$ ,  $p$  is visitable in  $\mathbf{A}$  if there exist a design  $\mathcal{D}$  in  $\mathbf{A}$  and a net  $\mathfrak{R}$  in  $\mathbf{A}^\perp$  such that  $p = \langle \mathcal{D} \leftarrow \mathfrak{R} \rangle$ .*
- *$V_{\mathbf{A}}$  is the set of visitable paths of  $\mathbf{A}$ .*

**Proposition 2.24** *Let  $\mathbf{A}$  be a behaviour,  $V_{\mathbf{A}} = \widetilde{V_{\mathbf{A}^\perp}}$ ,*

*Proof* By definition,  $p \in V_{\mathbf{A}}$  iff there exist a design  $\mathcal{D} \in \mathbf{A}$  and a net  $\mathfrak{R} \in \mathbf{A}^\perp$  such that  $p = \langle \mathcal{D} \leftarrow \mathfrak{R} \rangle$ , *i.e.*,  $\tilde{p} = \langle \mathfrak{R} \leftarrow \mathcal{D} \rangle$ . Thus, as  $\mathcal{D} \in \mathbf{A} = \mathbf{A}^{\perp\perp}$ ,  $\tilde{p} \in V_{\mathbf{A}^\perp}$ . Hence  $\widetilde{V_{\mathbf{A}}} \subset V_{\mathbf{A}^\perp}$ , *i.e.*,  $V_{\mathbf{A}} = \widetilde{V_{\mathbf{A}}} \subset \widetilde{V_{\mathbf{A}^\perp}}$ . Replacing  $\mathbf{A}$  by  $\mathbf{A}^\perp$ , we have also  $\widetilde{V_{\mathbf{A}^\perp}} \subset V_{\mathbf{A}}$ . So  $V_{\mathbf{A}} = \widetilde{V_{\mathbf{A}^\perp}}$ .

**Definition 2.25 (Incarnation)** *Let  $\mathbf{A}$  be a behaviour,  $\mathcal{D}$  be a design in  $\mathbf{A}$ . The design  $\mathcal{D}$  is material (or incarnated) in the behaviour  $\mathbf{A}$  if  $\mathcal{D}$  is minimal in  $\mathbf{A}$  with respect to inclusion. We denote by  $|\mathcal{D}|_{\mathbf{A}}$  the material design of  $\mathbf{A}$  that is included in  $\mathcal{D}$ . The incarnation  $|\mathbf{A}|$  of a behaviour  $\mathbf{A}$  is the set of its material designs.*

**Proposition 2.26** *Let  $\mathbf{A}$  be a behaviour,  $\mathcal{D}$  be a design in  $\mathbf{A}$ .  $|\mathcal{D}|_{\mathbf{A}} = \bigcup_{\mathfrak{R} \in \mathbf{A}^\perp} \ulcorner \langle \mathcal{D} \leftarrow \mathfrak{R} \rangle \urcorner$ .*

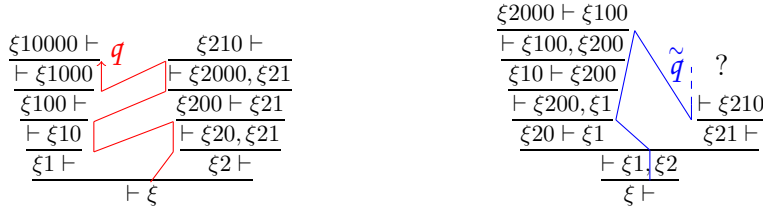


Figure 3: A path  $q$  in the shuffle of two designs the dual  $\tilde{q}$  of which is not a path.

*Proof* Remark that  $\bigcup_{\mathfrak{R} \in \mathbf{A}^\perp} \ulcorner \langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle \urcorner = \ulcorner \bigcup_{\mathfrak{R} \in \mathbf{A}^\perp} \langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle \urcorner$ . To obtain a design  $\mathfrak{D}_0$  strictly included in  $\bigcup_{\mathfrak{R} \in \mathbf{A}^\perp} \ulcorner \langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle \urcorner$ , we have to erase at least a chronicle  $\mathfrak{c}$  (and its extensions). But there is at least a path  $p_0 \in \bigcup_{\mathfrak{R} \in \mathbf{A}^\perp} \langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  such that  $\mathfrak{c} \in \ulcorner p_0 \urcorner$  hence if we denote by  $\mathfrak{R}_0$  a net such that  $p_0 = \langle \mathfrak{D} \leftarrow \mathfrak{R}_0 \rangle$  we have by linearity constraint on designs that  $\mathfrak{D}_0 \not\leq \mathfrak{R}_0$ . Hence  $\mathfrak{D}_0 \notin \mathbf{A}$ .

### 3 Visitable Paths and Logical Decomposition

In this section we relate MALL logical connectives, in fact  $\otimes$ ,  $\oplus$ ,  $\downarrow$ , with operations on visitable paths. Operations  $\oplus$  and  $\downarrow$  are quite immediate. The behaviour  $\mathbf{A} \oplus \mathbf{B}$ , is the union of the two behaviours  $\mathbf{A}$  and  $\mathbf{B}$ , hence visitable paths of the result should be the union of the two sets of visitable paths. The behaviour  $\downarrow \mathbf{A}$  is built by adding an identical action as root, hence visitable paths of the resulting behaviour should be built by adding this action as prefix of visitable paths of  $\mathbf{A}$ . As  $\otimes$  models a kind of concurrency, it is natural to consider that the set of visitable paths of a tensor should be in some way the shuffle of the sets of underlying behaviours. However belonging to the shuffle of visitable paths is not sufficient for being a visitable path in the tensor of two behaviours as shown in the following example.

**Example 3.1** Let us consider behaviours  $\mathbf{A} = \{\mathfrak{A}\}^{\perp\perp}$  and  $\mathbf{B} = \{\mathfrak{B}\}^{\perp\perp}$  where:

$$\mathfrak{A} = \frac{\frac{\frac{\frac{\xi 10000 \vdash}{\vdash \xi 1000}}{\xi 100 \vdash}}{\vdash \xi 10}}{\xi 1 \vdash} \vdash \xi \quad \text{and} \quad \mathfrak{B} = \frac{\frac{\frac{\xi 210 \vdash}{\vdash \xi 2000, \xi 21}}{\xi 200 \vdash \xi 21}}{\vdash \xi 20, \xi 21}}{\xi 2 \vdash} \vdash \xi$$

Let us consider the path  $q = (+, \xi, \{1, 2\})(-, \xi 2, \{0, 1\})(+, \xi 20, \{0\})(-, \xi 1, \{0\})(+, \xi 10, \{0\})(-, \xi 200, \{0\})(+, \xi 21, \{0\})(-, \xi 100, \{0\})(+, \xi 1000, \{0\})$ . We have that  $q$  is a path in  $\mathbf{A} \otimes \mathbf{B}$  as shown in Fig. 3 on the left (path in red). It is a shuffle of the two following paths

- $p_1 = (+, \xi, \{1, 2\})(-, \xi 2, \{0, 1\})(+, \xi 20, \{0\})(-, \xi 200, \{0\})(+, \xi 21, \{0\})$
- $p_2 = (+, \xi, \{1, 2\})(-, \xi 1, \{0\})(+, \xi 10, \{0\})(-, \xi 100, \{0\})(+, \xi 1000, \{0\})$

These two paths  $p_1$  and  $p_2$  are visitable respectively in  $\mathbf{A}$  and  $\mathbf{B}$ . However  $\tilde{q}$  is not a path. It suffices to prove that  $\ulcorner \tilde{q} \urcorner$  is not a design. In fact  $\ulcorner \tilde{q} \urcorner$  is not even a chronicle:  $\ulcorner \tilde{q} \urcorner = (-, \xi, \{1, 2\})(+, \xi 2, \{0, 1\})(-, \xi 21, \{0\})(+, \xi 100, \{0\})(-, \xi 1000, \{0\})\star$  and the justifier of  $\xi 100$  is not present in  $\ulcorner \tilde{q} \urcorner$ . This may also be noticed in figure below on the right:  $\tilde{q}$  defines in blue a path in an object that cannot be completed to give rise to a design with  $\tilde{q}$  fully present. Hence  $q \notin V_{\mathbf{A} \otimes \mathbf{B}}$ .

### 3.1 Extension of a base

Extension is a technical trick to simplify the presentation of results on the tensor  $\otimes$  operation. In short, the extension is applied to positive behaviours with a simple base  $\vdash \xi$ . The first action of designs of such a behaviour is necessarily of the form  $(+, \xi, I)$ . The extension to  $J$  is the set of designs where the first action is replaced by  $(+, \xi, I \cup J)$ , this operation is defined only when  $J$  does not overlap any such  $I$ . Such an operation does not change visitable paths. In fact, we conveniently present the extension with respect to a behaviour or to a set of paths and not only to a set of addresses  $J$ . We remind that two positive behaviours  $\mathbf{A}$  and  $\mathbf{B}$  are *alien* when the ramifications of the first actions of designs belonging to  $\mathbf{A}$  and the ramifications of the first actions of designs belonging to  $\mathbf{B}$  are pairwise disjoint.

**Definition 3.2 (Extension)** *Let  $\mathbf{A}$  and  $\mathbf{B}$  be alien positive behaviours of base  $\vdash \xi$ .*

- *The extension of  $\mathbf{A}$  with respect to  $\mathbf{B}$ , denoted by  $\mathbf{A}_{[\mathbf{B}]}$ , is the set of designs  $\mathfrak{D}$  such that either  $\mathfrak{D} = \mathfrak{D}\mathfrak{a}\mathfrak{i}$  or there exist designs  $\mathfrak{A} \in \mathbf{A}$  of first action  $(+, \xi, I)$  and  $\mathfrak{B} \in \mathbf{B}$  of first action  $(+, \xi, J)$  and  $\mathfrak{D}$  is obtained from  $\mathfrak{A}$  by replacing its first action by  $(+, \xi, I \cup J)$ .*
- *Let  $P$  (resp.  $Q$ ) be a set of paths of  $\mathbf{A}$  (resp.  $\mathbf{B}$ ), the extension of  $P$  with respect to  $Q$ , denoted by  $P_{[Q]}$ , is the set of paths  $p$  such that either  $p = \star$  or there exist paths  $r \in P$  of first action  $(+, \xi, I)$  and  $s \in Q$  of first action  $(+, \xi, J)$  and  $p$  is obtained from  $r$  by replacing its first action by  $(+, \xi, I \cup J)$ .*

**Lemma 3.3**  $\mathbf{A}_{[\mathbf{B}]}$  is a behaviour.  $V_{(\mathbf{A}_{[\mathbf{B}]})} = (V_{\mathbf{A}})_{[V_{\mathbf{B}}]}$ .

*Proof* It follows from the definition that  $p$  is a path of  $\mathbf{A}_{[\mathbf{B}]}$  iff  $p$  is either the daimon or of the form  $(+, \xi, I \cup J)q$  where  $(+, \xi, I)q$  is a path of  $\mathbf{A}$  and  $(+, \xi, J)$  is the first action of a design of  $\mathbf{B}$ . Hence also a path in  $(\mathbf{A}_{[\mathbf{B}]})^\perp$  is either empty or begins with  $(-, \xi, I \cup J)$ .

Let  $\mathfrak{G} \in |\mathbf{A}_{[\mathbf{B}]}^\perp|$ . For each  $\mathfrak{F} \in \mathbf{A}_{[\mathbf{B}]}^\perp$ , consider the path  $\langle \mathfrak{G} \leftarrow \mathfrak{F} \rangle$ : either it is the daimon or it is of the form  $(+, \xi, I \cup J)p_{\mathfrak{F}}$  where  $(+, \xi, I)$  (resp.  $(+, \xi, J)$ ) is a first action of a design of  $\mathbf{A}$  (resp.  $\mathbf{B}$ ). Remark that sequences  $(+, \xi, I \cup J)p_{\mathfrak{F}}$  are pairwise coherent paths. Hence it follows from proposition 2.26 that  $\mathfrak{G} = \ulcorner \mathfrak{D}\mathfrak{a}\mathfrak{i} \cup \bigcup_{\mathfrak{F}} (+, \xi, I \cup J)p_{\mathfrak{F}} \urcorner$ . We note now that the design  $\ulcorner \mathfrak{D}\mathfrak{a}\mathfrak{i} \cup \bigcup_{\mathfrak{F}} (+, \xi, I)p_{\mathfrak{F}} \urcorner$  is in the behaviour  $\mathbf{A}$ . Hence  $\mathfrak{G}$  is in  $\mathbf{A}_{[\mathbf{B}]}$ . Thus  $|\mathbf{A}_{[\mathbf{B}]}^\perp| \subset \mathbf{A}_{[\mathbf{B}]}$ . Note now that,  $\mathbf{A}$  being a behaviour, let  $\mathfrak{D} \in \mathbf{A}$ , if  $\mathfrak{E}$  is a design that includes  $\mathfrak{D}$  then  $\mathfrak{E} \in \mathbf{A}$ . Hence also let  $\mathfrak{D} \in \mathbf{A}_{[\mathbf{B}]}$ , if  $\mathfrak{E}$  is a design that includes  $\mathfrak{D}$  then  $\mathfrak{E} \in \mathbf{A}_{[\mathbf{B}]}$ . Thus  $\mathbf{A}_{[\mathbf{B}]}^\perp \subset \mathbf{A}_{[\mathbf{B}]}$ . It follows that  $\mathbf{A}_{[\mathbf{B}]}$  is a behaviour.

From the previous reasoning, we deduce easily that  $V_{(\mathbf{A}_{[\mathbf{B}]})} = (V_{\mathbf{A}})_{[V_{\mathbf{B}}]}$ .

### 3.2 Visitability for Linear Operations on Behaviours

The interpretation of Multiplicative-Additive Linear Logic (MALL) in the framework of Ludics is a real success. Not only are there full completeness and soundness results but completeness is *internal*: product and sum of two behaviours is a behaviour (without need for closing by bi-orthogonality the result). In the original paper of Girard [10], these results are also achieved for second-order, however we do not consider it in this paper. Before establishing properties concerning visitability (proposition 3.6 and theorem 3.7), we recall below the main linear operations on behaviours: multiplicative tensor ' $\otimes$ ', additive sum ' $\oplus$ ' and also the shift ' $\downarrow$ ' operation. The shift operation is required as the logics is polarized: it allows for switching from/to a positive behaviour to/from a negative behaviour. Dual operations are defined in a standard way:  $\mathbf{A} \wp \mathbf{B} = (\mathbf{A} \otimes \mathbf{B})^\perp$ ,  $\mathbf{A} \& \mathbf{B} = (\mathbf{A} \otimes \mathbf{B})^\perp$  and  $\uparrow \mathbf{A} = (\downarrow \mathbf{A})^\perp$ .

### Definition 3.4

- Let  $\mathbf{G}_k$  be a family of positive behaviours pairwise disjoint,  $\bigoplus_k \mathbf{G}_k = (\bigcup_k \mathbf{G}_k)^{\perp\perp}$
- Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two positive alien designs:
  - If  $\mathfrak{A}$  or  $\mathfrak{B}$  is  $\mathfrak{Dai}$ , then  $\mathfrak{A} \otimes \mathfrak{B} = \mathfrak{Dai}$ .
  - Otherwise  $\mathfrak{A} = (+, \xi, I)\mathfrak{A}'$  and  $\mathfrak{B} = (+, \xi, I)\mathfrak{B}'$  then  $\mathfrak{A} \otimes \mathfrak{B} = (+, \xi, I \cup J)(\mathfrak{A}' \cup \mathfrak{B}')$ .
- Let  $\mathbf{G}$  and  $\mathbf{H}$  be two positive alien behaviours,  $\mathbf{G} \otimes \mathbf{H} = \{\mathfrak{A} \otimes \mathfrak{B} ; \mathfrak{A} \in \mathbf{G}, \mathfrak{B} \in \mathbf{H}\}^{\perp\perp}$
- Let  $\mathbf{G}$  be a negative behaviour of base  $\xi i \vdash$ ,  $\downarrow \mathbf{G} = ((+, \xi, \{i\})\mathbf{G})^{\perp\perp}$ .

### Theorem 3.5 (internal completeness [10])

- Let  $K \neq \emptyset$ ,  $\bigoplus_{k \in K} \mathbf{G}_k = \bigcup_{k \in K} \mathbf{G}_k$
- A behaviour of positive base is always decomposable as a  $\bigoplus$  of connected behaviours.
- Let  $\mathbf{G}$  and  $\mathbf{H}$  be two alien positive behaviours,  $\mathbf{G} \otimes \mathbf{H} = \{\mathfrak{A} \otimes \mathfrak{B} ; \mathfrak{A} \in \mathbf{G}, \mathfrak{B} \in \mathbf{H}\}$ .
- Let  $\mathbf{G}$  be a negative behaviour of base  $\xi i \vdash$ ,  $\downarrow \mathbf{G} = \{\mathfrak{Dai}\} \cup (+, \xi, \{i\})\mathbf{G}$ .
- Ludics is fully sound and complete with respect to polarized multiplicative-additive Linear Logic.

**Proposition 3.6** Let  $\mathbf{A}$  be a negative behaviour, then  $V_{\downarrow \xi i \mathbf{A}} = \{\star\} \cup (+, \xi, \{i\})V_{\mathbf{A}}$ .  
 Let  $(\mathbf{A}_k)_{k \in K}$  be a family of pairwise disjoint positive behaviours with  $K \neq \emptyset$ , then  $V_{\bigoplus_{k \in K} \mathbf{A}_k} = \biguplus_{k \in K} V_{\mathbf{A}_k}$ .  
 Let  $\mathbf{A}$  and  $\mathbf{B}$  be alien positive behaviours, then  $V_{\mathbf{A} \otimes \mathbf{B}} = \{q ; \tilde{q} \text{ is a path and } q \in V_{\mathbf{A}[\mathbf{B}]} \uplus V_{\mathbf{B}[\mathbf{A}]}\}$ .

*Proof* Let  $\mathbf{A}$  be a negative behaviour with base  $\xi i \vdash$ .  $\downarrow_{\xi i} \mathbf{A}$  is a behaviour then  $\star$  is visitable in it. Let  $\mathfrak{C}$  be a design of  $(\downarrow_{\xi i} \mathbf{A})^\perp$  distinct from  $\mathfrak{Dai}$ . Note that the only first proper action of designs of  $\downarrow_{\xi i} \mathbf{A}$  is  $(+, \xi, \{i\})$ . Hence one can define  $\mathfrak{C}' = \{\mathfrak{c} ; (-, \xi, \{i\})\mathfrak{c} \in \mathfrak{C}\}$ . It is immediate that  $\mathfrak{C}'$  is a design. As normalization is deterministic,  $\mathfrak{C}' \in \mathbf{A}^\perp$ . Reversely, from a design  $\mathfrak{C}' \in \mathbf{A}^\perp$ , one builds a design  $\mathfrak{C} = (-, \xi, \{i\})\mathfrak{C}' \in (\downarrow_{\xi i} \mathbf{A})^\perp$ . Hence  $p$  is a visitable path of  $\downarrow_{\xi i} \mathbf{A}$  iff  $p = \star$  or  $p = (+, \xi, \{i\})p'$  where  $p'$  is a visitable path of  $\mathbf{A}$ .

Let  $(\mathbf{A}_k)_{k \in K}$  be a family of disjoint positive behaviours with base  $\vdash \xi$ .

- Let  $p \in V_{\bigoplus_{k \in K} \mathbf{A}_k}$ : there exist designs  $\mathfrak{D} \in \bigoplus_{k \in K} \mathbf{A}_k$  and  $\mathfrak{C} \in (\bigoplus_{k \in K} \mathbf{A}_k)^\perp$  such that  $p = \langle \mathfrak{D} \leftarrow \mathfrak{C} \rangle$ . By the disjunction property, ([10], Th. 11), we have that  $\bigoplus_{k \in K} \mathbf{A}_k = \biguplus_{k \in K} \mathbf{A}_k$ . Hence there exist  $k_0$  such that  $\mathfrak{D} \in \mathbf{A}_{k_0}$  and  $\mathfrak{C} \in \bigcap_{k \in K} \mathbf{A}_k^\perp$ , so in particular  $\mathfrak{C} \in \mathbf{A}_{k_0}^\perp$ . Thus  $p \in V_{\mathbf{A}_{k_0}} \subset \biguplus_{k \in K} V_{\mathbf{A}_k}$ .
- Let  $p \in \biguplus_{k \in K} V_{\mathbf{A}_k}$ . Hence there exist  $k_0 \in K$  such that  $p \in V_{\mathbf{A}_{k_0}}$ : there exist designs  $\mathfrak{D} \in \mathbf{A}_{k_0}$  and  $\mathfrak{C} \in \mathbf{A}_{k_0}^\perp$  such that  $p = \langle \mathfrak{D} \leftarrow \mathfrak{C} \rangle$ . As behaviours  $\mathbf{A}_k$  are pairwise disjoint, we can add to  $\mathfrak{C}$  (or replace) chronicles of the form  $(-, \xi, I)\star$  where  $I$  is in the directory of  $\mathbf{A}_k$ ,  $k \neq k_0$ , to get a design  $\mathfrak{C}'$ . Hence  $\mathfrak{C}' \in \bigcap_{k \in K} \mathbf{A}_k^\perp$  and  $p = \langle \mathfrak{D} \leftarrow \mathfrak{C}' \rangle$ . Thus  $p \in V_{\bigoplus_{k \in K} \mathbf{A}_k}$ .

Let  $\mathbf{A}$  and  $\mathbf{B}$  be alien positive behaviours with base  $\vdash \xi$ .

- Let  $p \in V_{\mathbf{A} \otimes \mathbf{B}}$ . As  $p$  is a visitable path,  $\tilde{p}$  is a path. Furthermore there exist designs  $\mathfrak{D} \in \mathbf{A} \otimes \mathbf{B}$  and  $\mathfrak{E} \in (\mathbf{A} \otimes \mathbf{B})^\perp$  such that  $p = \langle \mathfrak{D} \leftarrow \mathfrak{E} \rangle$ . Using the independence property ([10], Th. 20), there exist designs  $\mathfrak{D}_1 \in \mathbf{A}$  and  $\mathfrak{D}_2 \in \mathbf{B}$  such that  $\mathfrak{D} = \mathfrak{D}_1 \otimes \mathfrak{D}_2$ .

If  $p = \mathfrak{A}$ , then  $p \in V_{\mathbf{A}[\mathbf{B}]}$ . So let us consider the other cases, *i.e.*, designs  $\mathfrak{D}_1$  and  $\mathfrak{D}_2$  are distinct from  $\mathfrak{D}\mathfrak{a}\mathfrak{i}$ .

Let  $(+, \xi, I)$  (resp.  $(+, \xi, J)$ ) be the first action of  $\mathfrak{D}_1$  (resp.  $\mathfrak{D}_2$ ). Behaviours being alien, we have that  $I \cap J = \emptyset$ . Note that each action of  $p$  is either  $(+, \xi, I \cup J)$ , or an action of design  $\mathfrak{D}_1$  or an action of design  $\mathfrak{D}_2$ , the three cases being exclusive of one another.

Let  $p_1 := (+, \xi, I)q_1$  (resp.  $p_2 := (+, \xi, J)q_2$ ) where  $q_1$  (resp.  $q_2$ ) is the subsequence of  $p$  made of actions of  $\mathfrak{D}_1$  (resp.  $\mathfrak{D}_2$ ). Following the adjunction theorem ([10], Th. 14), we have that  $\mathfrak{D}_1 \perp (\mathfrak{E})\mathfrak{D}_2$  with  $(\mathfrak{E})\mathfrak{D}_2$  independent from  $\mathfrak{D}_1$ , *i.e.*,  $(\mathfrak{E})\mathfrak{D}_2 \in \mathbf{A}^\perp$ . Similarly,  $(\mathfrak{E})\mathfrak{D}_1 \in \mathbf{B}^\perp$ . Note that  $p_1 = \langle \mathfrak{D}_1 \leftarrow (\mathfrak{E})\mathfrak{D}_2 \rangle \in V_{\mathbf{A}}$  and  $p_2 = \langle \mathfrak{D}_2 \leftarrow (\mathfrak{E})\mathfrak{D}_1 \rangle \in V_{\mathbf{B}}$ . Finally remark that  $p \in (+, \xi, I \cup J)q_1 \sqcup (+, \xi, I \cup J)q_2$  (as jumps are on negative actions). Hence  $p \in V_{\mathbf{A}[\mathbf{B}]} \sqcup V_{\mathbf{B}[\mathbf{A}]}$ .

- Let  $p \in V_{\mathbf{A}}$ ,  $p' \in V_{\mathbf{B}}$ . If one of the two paths  $p$  or  $p'$  is the daimon, then their shuffle is the daimon, hence in  $V_{\mathbf{A} \otimes \mathbf{B}}$ . Otherwise,  $p = (+, \xi, I)p_1$  and  $p' = (+, \xi, J)p'_1$ . As  $p \in V_{\mathbf{A}}$ , there exist designs  $\mathfrak{D} \in \mathbf{A}$  and  $\mathfrak{E} \in \mathbf{A}^\perp$  such that  $p = \langle \mathfrak{D} \leftarrow \mathfrak{E} \rangle$ . Similarly there exist designs  $\mathfrak{D}' \in \mathbf{B}$  and  $\mathfrak{E}' \in \mathbf{B}^\perp$  such that  $p' = \langle \mathfrak{D}' \leftarrow \mathfrak{E}' \rangle$ . Let  $q \in (+, \xi, I \cup J)p_1 \sqcup (+, \xi, I \cup J)p'_1$  such that  $\tilde{q}$  is a path. We will prove that  $q$  which is a path of  $\mathfrak{D} \otimes \mathfrak{D}'$  is visitable in  $\mathbf{A} \otimes \mathbf{B}$ . Let us consider the design  $\mathfrak{G} := \pi^{\sim} \tilde{q}^{\neg c}$ . We prove by contradiction that  $\mathfrak{G} \in (\mathbf{A} \otimes \mathbf{B})^\perp$ . Let  $\mathfrak{F} \in \mathbf{A}$  and  $\mathfrak{F}' \in \mathbf{B}$ . Suppose that  $\mathfrak{F} \otimes \mathfrak{F}' \not\perp \mathfrak{G}$ . Notice that, since a slice of  $\mathfrak{G}$  is finite, it is not possible that the interaction between  $\mathfrak{G}$  and  $\mathfrak{F} \otimes \mathfrak{F}'$  would be infinite. Let  $r$  be the sequence of actions of  $\mathfrak{F} \otimes \mathfrak{F}'$  used during the normalization before divergence. We consider the two cases of non-convergence, due to the fact that the next (proper positive) action in one of the designs has no dual in the other design:
  - action  $\kappa^+$  in  $\mathfrak{F}$  or  $\mathfrak{F}'$  and  $\overline{\kappa^+} \notin \mathfrak{G}$ : this contradicts the construction of  $\mathfrak{G}$ .
  - action  $\kappa^+$  in  $\mathfrak{G}$  and  $\overline{\kappa^+} \notin \mathfrak{F} \otimes \mathfrak{F}'$ . By construction of  $\mathfrak{G}$ ,  $\kappa^+ \in \pi^{\sim} \tilde{q}^{\neg c}$ . Hence  $\overline{\kappa^+} \in \pi^{\sim} \tilde{q}^{\neg c}$ . Notice that  $\pi^{\sim} \tilde{q}^{\neg c}$  is a slice, hence  $\kappa^+$  appears only once in  $\pi^{\sim} \tilde{q}^{\neg c}$ . Thus, w.l.o.g.,  $\overline{\kappa^+}$  appears in  $\pi^{\sim} \tilde{p}^{\neg c}$ . Let  $r_1$  the subsequence of  $r$  followed in  $\mathfrak{F}$ . Remark that  $r_1 \in \pi^{\sim} \tilde{p}^{\neg c}$ , hence also  $r_1 \kappa^+ \in \pi^{\sim} \tilde{p}^{\neg c}$ . The design  $\pi^{\sim} \tilde{p}^{\neg c}$  is included in  $\mathfrak{E}$ , then by proposition 2.22, the design  $\pi^{\sim} \tilde{p}^{\neg c}$  belongs to  $\mathbf{A}^\perp$ , then,  $\pi^{\sim} \tilde{p}^{\neg c} \perp \mathfrak{F}$ , hence a contradiction.
 Thus  $\mathfrak{F} \otimes \mathfrak{F}' \perp \mathfrak{G}$ . Finally, by construction, we have that  $q = \langle \mathfrak{D} \otimes \mathfrak{D}' \leftarrow \mathfrak{G} \rangle$ . So  $q \in V_{\mathbf{A} \otimes \mathbf{B}}$ .

**Theorem 3.7** Let  $\mathbf{P}$  be a positive behaviour,  $\mathbf{N}$  be a negative behaviour,  $(\mathbf{P}_k)_{k \in K}$  be a family of pairwise disjoint positive behaviours with  $K \neq \emptyset$ ,  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  be positive behaviours.

- $V_{\mathbf{P}} = \{\mathfrak{D}\mathfrak{a}\mathfrak{i}\} \cup (+, \xi, \{i\})V_{\mathbf{N}}$  iff  $\mathbf{P} = \downarrow_{\xi i} \mathbf{N}$ .
- $V_{\mathbf{P}} = \biguplus_{k \in K} V_{\mathbf{P}_k}$  iff  $\mathbf{P} = \bigoplus_{k \in K} \mathbf{P}_k$ .
- $V_{\mathbf{P}} = \{q ; \tilde{q} \text{ is a path and } q \in V_{\mathbf{Q}_1[\mathbf{Q}_2]} \sqcup V_{\mathbf{Q}_2[\mathbf{Q}_1]}\}$  iff  $\mathbf{P} = \mathbf{Q}_1 \otimes \mathbf{Q}_2$ .

*Proof* The properties follow from propositions 3.6 and 2.24.



## 4 Regular Behaviours

We prove in following sections that *regularity* as it is defined below is the main concept for a behaviour to be the denotation of a MALL formula. The only other constraint that is necessary concerns the finite structure of such a formula. This last constraint is specified in the next section. In a few words, a regular behaviour is a behaviour such that all ‘reasonable’ paths are visitable in this behaviour. ‘Reasonable’ means not only reversible but also made of actions that are present in the incarnation of a behaviour: such a path is called *regular for* this behaviour. Obviously, a visitable path for a behaviour is a regular path for this behaviour.

**Definition 4.1** Let  $\mathbf{A}$  be a behaviour, let  $\kappa$  be a proper action of a chronicle  $\mathfrak{c}$  of a design of the incarnation  $|\mathbf{A}|$ , the sequence  $\langle \kappa \rangle_{\mathfrak{c}}$  of justifiers of  $\kappa$  in  $\mathfrak{c}$  until an initial action with base the one of  $\mathbf{A}$  is called a **trivial chronicle** for  $\mathbf{A}$ .

It is worth noticing that a trivial chronicle is a chronicle such that each action is justified by the immediate previous one in the sequence. For ease of reading, we may note  $\langle \kappa \rangle$  making implicit the chronicle (and the design) from which the sequence is extracted.

**Definition 4.2** *A reversible path  $p$  is regular for a behaviour  $\mathbf{A}$  if  $p$  has the same base as  $\mathbf{A}$  and if, for each proper action  $\kappa$  of  $p$ , the subsequence of justifiers of  $\kappa$  in  $p$  is a trivial chronicle for  $\mathbf{A}$ .*

A behaviour  $\mathbf{A}$  is **regular** when each regular path for  $\mathbf{A}$  is visitable in  $\mathbf{A}$ , i.e.,  $R_{\mathbf{A}} \subset V_{\mathbf{A}}$ .

Remark that a trivial chronicle for a behaviour  $\mathbf{A}$  is regular for  $\mathbf{A}$ , hence trivial chronicles for a behaviour are visitable as soon as this behaviour is regular. Remark also that a path  $p$  in a design of the incarnation of a behaviour is regular for this behaviour: the set of actions of  $p$  may always be viewed as the union of the sequences of immediate justification for each proper action, and these sequences are trivial chronicles as they are subsequences of a path in the incarnation! Thus in particular we always have  $V_{\mathbf{A}} \subset R_{\mathbf{A}}$ . However a path regular for a behaviour may not be visitable for this behaviour: in Example 3.1, the path  $(+, \xi, \{2\})(-, \xi 2, \{0, 1\})(+, \xi 21, \{0\})$  is regular for  $\mathbf{B}$  but is not visitable in  $\mathbf{B}$ . We give below another example of a regular path that is not visitable.

**Example 4.3** *Let us consider the behaviour  $\mathbf{C} = \{\mathfrak{C}\}^{\perp\perp}$  where*

$$\begin{array}{c} \frac{\xi 00 \vdash}{\vdash \xi 0} \\ \frac{\sigma 0 \vdash \xi 0}{\vdash \xi 0, \sigma} \\ \text{C} = \frac{}{\xi \vdash \sigma} \end{array} \quad \begin{array}{c} \frac{}{\vdash \xi 11111} * \\ \frac{\xi 1111 \vdash}{\vdash \xi 111} \\ \frac{\xi 11 \vdash}{\vdash \xi 1} \end{array}$$

The trivial chronicles are the three following ones and their prefixes:  $\langle(+, \xi 0, \{0\})\rangle$ ,  $\langle(-, \xi 1111, \{1\})\rangle$ ,  $\langle(-, \sigma 0, \emptyset)\rangle$ . The following sequence is a regular path for  $\mathbf{C}$  but is not visitable in  $\mathbf{C}$ :  $(-, \xi, \{1\})(+, \xi 1, \{1\})(-, \xi 11, \{1\})(+, \xi 111, \{1\})(-, \xi 1111, \{1\})(+, \sigma, \{0\})$

We prove in the following that regularity is stable with respect to linear logical operations, *i.e.*,  $\cdot^\perp$ ,  $\otimes$ ,  $\oplus$  and  $\downarrow$ .

**Lemma 4.4** *Let  $\mathbf{A}$  be a regular behaviour, then the behaviour  $\mathbf{A}^\perp$  is regular.*

*Proof* Following proposition 2.24, we know that an action in  $|\mathbf{A}^\perp|$  is either  $\star$  or the dual of an action in  $|\mathbf{A}|$ . Let  $\langle \kappa \rangle$  be a trivial chronicle for  $\mathbf{A}^\perp$  then there exists a path  $q$  visitable in  $\mathbf{A}^\perp$  such that  $\langle \kappa \rangle$  is a subsequence of  $q$ . The path  $\tilde{q}$  is visitable in  $\mathbf{A}$  and the sequence  $\overline{\langle \kappa \rangle}$  of opposite polarities is a subsequence of  $\tilde{q}$  and made of actions of immediate justification, hence  $\overline{\langle \kappa \rangle}$  is a trivial chronicle for  $\mathbf{A}$ . Let  $p \in R_{\mathbf{A}^\perp}$ , from the previous remark it follows that  $\tilde{p}$  is a reversible path made of actions in trivial chronicles of  $\mathbf{A}$  or the daimon. Hence  $\tilde{p}$  is regular for  $\mathbf{A}$ . Thus  $\tilde{p} \in V_{\mathbf{A}}$  thus  $p \in \widetilde{V_{\mathbf{A}}} = V_{\mathbf{A}^\perp}$ .

**Lemma 4.5** *Let  $\mathbf{A}$  and  $\mathbf{B}$  be regular alien positive behaviours distinct from  $\mathbf{0}$ , then the behaviour  $\mathbf{A} \otimes \mathbf{B}$  is regular.*

*Proof* Let  $p$  be a path of  $R_{\mathbf{A} \otimes \mathbf{B}}$ . Obviously a positive-ended prefix of  $p$  is in  $R_{\mathbf{A} \otimes \mathbf{B}}$ . Recall also that  $p$  is of the form  $\kappa_0^+ \kappa_1^- \kappa_1^+ \dots \kappa_n^- \kappa_n^+$ . We prove that  $p \in V_{\mathbf{A} \otimes \mathbf{B}}$  by induction on  $n$ .

- The result is immediate when  $n = 0$ .
- Suppose the property satisfied for  $n$ , and  $p = p_1 \kappa_{n+1}^- \kappa_{n+1}^+$ . By induction hypothesis  $p_1 \in V_{\mathbf{A} \otimes \mathbf{B}}$ . It follows from proposition 3.6 that there exist paths  $a \in V_{\mathbf{A}}$  and  $b \in V_{\mathbf{B}}$  such that  $p_1 \in a \sqcup b$ . Note that for each positive action  $\kappa^+$  such that  $p_1' \kappa^+$  is a prefix of  $p_1$ , if  $\kappa^+$  occurs in  $a$  then  $\lceil p_1' \rceil \kappa^+$  is a subsequence of  $a$  except the first action whose ramification is an extension of the first action of  $a$  (it is a consequence of the definitions of shuffle and view operations). Furthermore  $\kappa_{n+1}^-$  and  $\kappa_{n+1}^+$  are actions in the incarnation of  $\mathbf{A} \otimes \mathbf{B}$  hence in visitable paths of  $\mathbf{A}_{[\mathbf{B}]}$  or  $\mathbf{B}_{[\mathbf{A}]}$ . Without loss of generality, we can suppose that  $\kappa_{n+1}^-$  occurs in a visitable path of  $\mathbf{A}_{[\mathbf{B}]}$ . Let  $\kappa^+$  be the justifier of  $\kappa_{n+1}^-$  in  $p$  and  $p_1' \kappa^+$  be the prefix of  $p$  ending with  $\kappa^+$ . We have that  $\kappa^+$  is in  $a$  as  $\kappa_{n+1}^-$  is in  $a$ . Furthermore  $\lceil p \rceil = \lceil p_1' \rceil \kappa^+ \kappa_{n+1}^- \kappa_{n+1}^+$ . Hence the justifier of  $\kappa_{n+1}^+$  is in  $a$ . Thus  $\kappa_{n+1}^+$  occurs in  $\mathbf{A}$  as the behaviours  $\mathbf{A}$  and  $\mathbf{B}$  are alien. It follows that  $\kappa_{n+1}^+$  occurs in a visitable path of  $\mathbf{A}_{[\mathbf{B}]}$ .

From the previous elements we have also that  $a \kappa_{n+1}^- \kappa_{n+1}^+$  is a path. Let us remark that  $p \in a \kappa_{n+1}^- \kappa_{n+1}^+ \sqcup b$  and  $p$  is reversible. Hence by lemma 2.18 we have in particular that  $a \kappa_{n+1}^- \kappa_{n+1}^+$  is a reversible path, hence is in  $R_{\mathbf{A}_{[\mathbf{B}]}}$ . It follows that  $a \kappa_{n+1}^- \kappa_{n+1}^+ \in V_{\mathbf{A}_{[\mathbf{B}]}}$ . Hence by proposition 3.6,  $p \in V_{\mathbf{A} \otimes \mathbf{B}}$ .

**Lemma 4.6** *Let  $\mathbf{A}$  and  $\mathbf{B}$  be alien positive behaviours distinct from  $\mathbf{0}$ , if  $\mathbf{A} \otimes \mathbf{B}$  is regular then  $\mathbf{A}$  and  $\mathbf{B}$  are regular.*

*Proof* Note that  $R_{\mathbf{A}_{[\mathbf{B}]}} \subset R_{\mathbf{A} \otimes \mathbf{B}}$ . Thus if  $p \in R_{\mathbf{A}_{[\mathbf{B}]}}$  then  $p \in V_{\mathbf{A} \otimes \mathbf{B}}$ . Hence  $p$  is a shuffle of a visitable path in  $\mathbf{A}$  and a visitable path in  $\mathbf{B}$ . As behaviours  $\mathbf{A}$  and  $\mathbf{B}$  are alien, they have no action in common except perhaps the daimon. Thus  $p$  is a visitable path in  $\mathbf{A}_{[\mathbf{B}]}$ . Let  $q \in R_{\mathbf{A}}$ , we can define a path  $q'$  extending the first action of  $q$  by the ramification of a first action of a design in  $\mathbf{B}$ . Thus  $q' \in R_{\mathbf{A}_{[\mathbf{B}]}}$ , so  $q' \in V_{\mathbf{A}_{[\mathbf{B}]}}$ . It follows that  $q \in V_{\mathbf{A}}$ . So  $\mathbf{A}$  is regular. Similarly,  $\mathbf{B}$  is regular.

**Proposition 4.7** *Behaviours  $\mathbf{0}$  and  $\mathbf{1}$  are regular.*

*Let  $\mathbf{A}$  be a negative behaviour,  $\mathbf{A}$  is regular iff  $\downarrow \mathbf{A}$  is regular.*

*Let  $(\mathbf{A}_k)_{k \in K}$  be a family of pairwise disjoint positive behaviours where  $K \neq \emptyset$ ,  $\mathbf{A}_k$  is regular for all  $k \in K$  iff  $\bigoplus_{k \in K} \mathbf{A}_k$  is regular.*

*Let  $\mathbf{A}$  and  $\mathbf{B}$  be alien positive behaviours distinct from  $\mathbf{0}$ ,  $\mathbf{A}$  and  $\mathbf{B}$  are regular iff  $\mathbf{A} \otimes \mathbf{B}$  is regular.*

*Proof*

- $R_0 = V_0 = \{\mathfrak{x}\}$ , hence  $\mathbf{0}$  is regular.
- $R_1 = V_1 = \{(+, \xi, \emptyset), \mathfrak{x}\}$ , hence  $\mathbf{1}$  is regular.
- Let  $\mathbf{A}$  be a negative behaviour of base  $\xi \vdash$ , we have that  $R_{\downarrow \xi} \mathbf{A} = (+, \xi, \{i\})R_{\mathbf{A}} \cup \{\mathfrak{x}\}$  and  $V_{\downarrow \xi} \mathbf{A} = (+, \xi, \{i\})V_{\mathbf{A}} \cup \{\mathfrak{x}\}$ . The result follows.
- Note first that the set of regular paths for  $\bigoplus_{k \in K} \mathbf{A}_k$  is exactly the disjoint union on  $K$  of the set of regular paths for  $\mathbf{A}_k$ . This follows immediately from the fact that the slices of  $|\bigoplus_{k \in K} \mathbf{A}_k|$  are exactly the disjoint union on  $K$  of the slices of  $|\mathbf{A}_k|$ . Then the result follows immediately from proposition 3.6.
- Let  $\mathbf{A}$  and  $\mathbf{B}$  be alien positive behaviours of base  $\vdash \xi$ : One direction is proved in lemma 4.5, the other in lemma 4.6.

The following proposition gives a more constructive presentation of regularity.

**Definition 4.8** *Let  $\mathbf{A}$  be a behaviour.  $\mathbf{A}$  is called **shuffle-regular** if all positive-ended chronicles of designs of  $|\mathbf{A}|$  and  $|\mathbf{A}^\perp|$  are visitable and  $V_{\mathbf{A}}$  and  $V_{\mathbf{A}^\perp}$  are stable by reversible shuffle (that is, all reversible paths in material designs of  $\mathbf{A}$  and  $\mathbf{A}^\perp$  are visitable).*

**Proposition 4.9** *Let  $\mathbf{A}$  be a behaviour of simple base, i.e., either  $\vdash \xi$  or  $\xi \vdash$ .  $\mathbf{A}$  is regular iff  $\mathbf{A}$  is shuffle-regular.*

*Proof* Let  $\mathbf{A}$  be a behaviour.

- Suppose that  $\mathbf{A}$  is regular.  
Let  $\mathfrak{c}$  be a positive-ended chronicle of a design  $\mathfrak{D}$  belonging to  $|\mathbf{A}|$ . A chronicle is reversible and in the incarnation hence  $\mathfrak{c}$  is a regular path for  $\mathbf{A}$  and then  $\mathfrak{c}$  is visitable.  
Let  $p$  and  $q$  be two visitable paths of  $\mathbf{A}$  and let  $r$  be a reversible path belonging to  $p \sqcup q$ . Remark that one may extract from  $r$  sequences of immediate justification for each action in  $r$ , and these sequences are either in  $p$  or  $q$ , thus are trivial chronicles. It follows that  $r$  is regular for  $\mathbf{A}$  hence visitable in  $\mathbf{A}$ . Finally it follows from lemma 4.4 and the previous reasoning that all positive-ended chronicles of designs of  $|\mathbf{A}^\perp|$  are visitable and  $V_{\mathbf{A}^\perp}$  is stable by reversible shuffle.
- Suppose that  $\mathbf{A}$  is shuffle-regular. Let  $p$  be a path, in the following we set  $n_p$  the number of positive actions of  $p$  justified each by a negative action that justifies at least two positive actions in  $p$ . In other words,  $n_p$  “measures” the distance to a trivial chronicle.
  - We first show that if a positive-ended chronicle  $\mathfrak{c}$  is a trivial chronicle for  $\mathbf{A}$  then  $\mathfrak{c}$  is visitable. By definition of what is a trivial chronicle, there is a design  $\mathfrak{D} \in |\mathbf{A}|$  and a chronicle  $\mathfrak{c} \in \mathfrak{D}$  such that  $\mathfrak{c}$  is a subsequence of  $\mathfrak{d}$ : all actions of  $\mathfrak{c}$  occur in  $\mathfrak{d}$  in the same order and  $\mathfrak{c}$  and  $\mathfrak{d}$  have the same positive last action. Let us notice that, in a trivial chronicle, each negative action justifies exactly one positive action. We show that  $\mathfrak{c}$  is visitable by induction on  $n_{\mathfrak{d}}$ .
    - \* If  $n_{\mathfrak{d}} = 0$  then  $\mathfrak{c} = \mathfrak{d}$  as the base is simple. Hence  $\mathfrak{c}$  is visitable as  $\mathbf{A}$  is shuffle-regular.

- \* Otherwise, the chronicle  $\mathfrak{d}$  may be written  $w_0\kappa^-w_1\kappa^+w_2$  such that  $\kappa^-$  justifies  $\kappa^+$  and  $\kappa^+w_2$  is the longest suffix of  $\mathfrak{d}$  made of actions such that each action justifies the following in the sequence (or the following is a daimon), hence  $\kappa^-$  justifies at least two positive actions in  $\mathfrak{d}$ . As  $\mathfrak{d} \in V_{\mathbf{A}}$ , we also have  $\tilde{\mathfrak{d}} \in V_{\mathbf{A}^\perp}$ . Note that  $\overline{w_0\kappa^-}\kappa^+w_2$  is a chronicle hence a path. Note also that  $\ulcorner \tilde{\mathfrak{d}} \urcorner = \ulcorner \overline{w_0\kappa^-} \urcorner \ulcorner \kappa^+w_2 \urcorner$  hence as  $\overline{w_0\kappa^-}$  is a path and  $\kappa^+$  is justified by  $\kappa^-$  we have that  $\overline{w_0\kappa^-}\kappa^+w_2$  is a path. Let us note  $\mathfrak{d}' = w_0\kappa^-\kappa^+w_2$ . The completed design  $\ulcorner \tilde{\mathfrak{d}} \urcorner^c$  is in the behaviour  $\mathbf{A}$  (proposition 2.22). Hence there exists a subdesign  $\mathfrak{E} \subset \ulcorner \tilde{\mathfrak{d}} \urcorner^c$  such that  $\mathfrak{E} \in |\mathbf{A}|$ . As  $\tilde{\mathfrak{d}} \in V_{\mathbf{A}}$ , we have necessarily that  $\ulcorner \tilde{\mathfrak{d}} \urcorner \subset \mathfrak{E}$ , hence also that the reversible path  $\tilde{\mathfrak{d}}'$  is a path of  $\mathfrak{E}$ . As the behaviour  $\mathbf{A}$  is shuffle-regular,  $\tilde{\mathfrak{d}}'$  is visitable in  $\mathbf{A}^\perp$ , hence  $\mathfrak{d}'$  is visitable in  $\mathbf{A}$ . Furthermore, the trivial chronicle  $\mathfrak{c}$  is a subsequence of  $\mathfrak{d}'$  and  $n_{\mathfrak{d}'} \leq n_{\mathfrak{d}} - 1$ . So by induction hypothesis, the chronicle  $\mathfrak{c}$  is visitable in  $\mathbf{A}$ .
- We show now that if a chronicle  $\mathfrak{c}$  is regular for  $\mathbf{A}$  then  $\mathfrak{c}$  is visitable in  $\mathbf{A}$ . The proof is done by induction on  $n_{\mathfrak{c}}$ .
  - \* If  $n_{\mathfrak{c}} = 0$  then  $\mathfrak{c}$  is a trivial chronicle for  $\mathbf{A}$  thus it follows from the previous item that  $\mathfrak{c}$  is visitable in  $\mathbf{A}$ .
  - \* Otherwise, the chronicle  $\mathfrak{c}$  may be written  $w_0\kappa^-w_1\kappa^+w_2$  such that  $\kappa^-$  justifies  $\kappa^+$  and  $\kappa^+w_2$  is the longest suffix of  $\mathfrak{c}$  made of actions such that each justifies the following in the sequence (or the following is a daimon), hence  $\kappa^-$  justifies at least two positive actions in  $\mathfrak{c}$ . Let us note  $\mathfrak{c}_1 = w_0\kappa^-\kappa^+w_2$  and  $\mathfrak{c}_2 = w_0\kappa^-w_1\mathfrak{x}$ . Note that the sequences  $\mathfrak{c}_1$  and  $\mathfrak{c}_2$  are chronicles hence reversible, furthermore regular as subsequences of a regular chronicle. Moreover both  $n_{\mathfrak{c}_1}$  and  $n_{\mathfrak{c}_2}$  are strictly less than  $n_{\mathfrak{c}}$ . Thus by induction hypothesis, these two chronicles are visitable in  $\mathbf{A}$ . Finally we note that the path  $\tilde{\mathfrak{c}}$  belongs to  $\tilde{\mathfrak{c}}_1 \sqcup \tilde{\mathfrak{c}}_2$ . So, as  $\mathbf{A}$  is a regular behaviour,  $\tilde{\mathfrak{c}}$  is visitable in  $\mathbf{A}^\perp$ , thus  $\mathfrak{c}$  is visitable in  $\mathbf{A}$ .
- Let  $p$  be a regular path for  $\mathbf{A}$ . From proposition 2.12, we know that  $p$  belongs to the shuffle of some of its views. Furthermore each view is a regular chronicle, hence visitable in  $\mathbf{A}$  by the previous item. Thus, as the behaviour  $\mathbf{A}$  is regular,  $p$  is visitable in  $\mathbf{A}$ .

**Example 4.10** We give below four characteristic examples of behaviours that show why regularity is useful for understanding the structure of a behaviour. In particular, we consider the four following constraints: to have chronicles in the incarnation that are visitable, to have stability of the shuffle property, and to satisfy the two constraints on the dual. In each three last cases, we can remark that the behaviours are positive but neither a shift, nor a tensor or a plus of two behaviours: the three last behaviours considered below are not interpretations of (composed) formulas of Linear Logic.

- Let us consider the behaviour  $\mathbf{E} = \{\mathfrak{E}\}^{\perp\perp}$  where the design  $\mathfrak{E}$  is given below on the left. The dual behaviour  $\mathbf{E}^\perp = \{\mathfrak{E}', \mathfrak{E}''\}^{\perp\perp}$  where the designs  $\mathfrak{E}'$  and  $\mathfrak{E}''$  are given below on the right.

$$\begin{array}{ccc}
\mathfrak{E} = \frac{\frac{\frac{\xi 100 \vdash}{\vdash \xi 10} \quad \frac{\xi 200 \vdash}{\vdash \xi 20}}{\xi 1 \vdash \quad \xi 2 \vdash}}{\vdash \xi} & \mathfrak{E}' = \frac{\frac{\frac{\frac{\vdash \xi 100, \xi 200}{\xi 20 \vdash \xi 100}}{\vdash \xi 2, \xi 100}}{\xi 10 \vdash \xi 2}}{\vdash \xi 1, \xi 2}}{\xi \vdash} & \mathfrak{E}'' = \frac{\frac{\frac{\frac{\vdash \xi 100, \xi 200}{\xi 10 \vdash \xi 200}}{\vdash \xi 1, \xi 200}}{\xi 20 \vdash \xi 1}}{\vdash \xi 1, \xi 2}}{\xi \vdash}
\end{array}$$

The set  $V_{\mathbf{E}}$  is easily computed: it is defined as the positive-prefix closed and daimon-prefix closed set given by the two following paths:

$$\begin{aligned} &(+, \xi, \{1, 2\})(-, \xi 1, 0)(+, \xi 10, 0)(-, \xi 2, 0)(+, \xi 20, 0), \\ &(+, \xi, \{1, 2\})(-, \xi 2, 0)(+, \xi 20, 0)(-, \xi 1, 0)(+, \xi 10, 0). \end{aligned}$$

The set  $V_{\mathbf{E}^\perp}$  is exactly the set of dual paths, i.e., the positive-prefix closed and daimon-prefix closed set given by the two following paths:

$$\begin{aligned} &(-, \xi, \{1, 2\})(+, \xi 1, 0)(-, \xi 10, 0)(+, \xi 2, 0)(-, \xi 20, 0)\star, \\ &(-, \xi, \{1, 2\})(+, \xi 2, 0)(-, \xi 20, 0)(+, \xi 1, 0)(-, \xi 10, 0)\star. \end{aligned}$$

It is immediate to prove that the behaviour  $\mathbf{E}$  is regular. Moreover the behaviour  $\mathbf{E}$  is fully decomposable with respect to operations of Linear Logic. As a matter of fact, we have that  $\mathbf{E} = (\downarrow_{\xi 1} \uparrow_{\xi 10} \mathbf{1}_{\xi 100}) \otimes (\downarrow_{\xi 2} \uparrow_{\xi 20} \mathbf{1}_{\xi 200})$  where we note  $\mathbf{1}_{\xi i}$  the behaviour obtained as the closure by bi-orthogonality of the following design:

$$\frac{\xi i \vdash}{\vdash \xi}$$

- Let us now consider the behaviour  $\mathbf{F} = \{\mathfrak{E}, \mathfrak{F}\}^{\perp\perp}$  where the designs  $\mathfrak{E}$  and  $\mathfrak{F}$  are given below on the left. The dual behaviour  $\mathbf{F}^\perp = \{\mathfrak{F}'\}^{\perp\perp}$  where the design  $\mathfrak{F}'$  is given below on the right.

$$\begin{aligned} \mathfrak{E} &= \frac{\frac{\xi 100 \vdash}{\vdash \xi 10} \quad \frac{\xi 200 \vdash}{\vdash \xi 20}}{\frac{\xi 1 \vdash}{\vdash \xi} \quad \frac{\xi 2 \vdash}{\vdash \xi}} & \mathfrak{F} &= \frac{\frac{\vdash \xi 10}{\xi 1 \vdash} \quad \xi 2 \vdash}{\vdash \xi} \star & \mathfrak{F}' &= \frac{\frac{\frac{\vdash \xi 100, \xi 200}{\xi 20 \vdash \xi 100} \star}{\vdash \xi 2, \xi 100}}{\frac{\xi 10 \vdash \xi 2}{\vdash \xi 1, \xi 2}} \star \end{aligned}$$

The set  $V_{\mathbf{F}}$  is easily computed: it is defined as the positive-prefix closed and daimon-prefix closed set given by the path  $p = (+, \xi, \{1, 2\})(-, \xi 1, 0)(+, \xi 10, 0)(-, \xi 2, 0)(+, \xi 20, 0)$ .

The behaviour  $\mathbf{F}$  satisfies the following properties:  $V_{\mathbf{F}}$  and  $V_{\mathbf{F}^\perp}$  are stable by shuffle and  $C_{\mathbf{F}^\perp} \subset V_{\mathbf{F}^\perp}$ . But  $C_{\mathbf{F}} \not\subset V_{\mathbf{F}}$ . Hence  $\mathbf{F}$  is not regular.

- Let us consider the behaviour  $\mathbf{G} = \{\mathfrak{E}, \mathfrak{G}\}^{\perp\perp}$  where the designs  $\mathfrak{E}$  and  $\mathfrak{G}$  are given below on the first line. The dual behaviour is  $\mathbf{G}^\perp = \{\mathfrak{G}', \mathfrak{G}''\}^{\perp\perp}$  where the designs  $\mathfrak{G}'$  and  $\mathfrak{G}''$  are given below on the second line.

$$\begin{aligned} \mathfrak{E} &= \frac{\frac{\xi 100 \vdash}{\vdash \xi 10} \quad \frac{\xi 200 \vdash}{\vdash \xi 20}}{\frac{\xi 1 \vdash}{\vdash \xi} \quad \frac{\xi 2 \vdash}{\vdash \xi}} & \mathfrak{G} &= \frac{\frac{\xi 101 \vdash}{\vdash \xi 10} \quad \frac{\xi 201 \vdash}{\vdash \xi 20}}{\frac{\xi 1 \vdash}{\vdash \xi} \quad \frac{\xi 2 \vdash}{\vdash \xi}} \\ \mathfrak{G}' &= \frac{\frac{\frac{\vdash \xi 100, \xi 200}{\xi 20 \vdash \xi 100} \star \quad \frac{\vdash \xi 101, \xi 201}{\xi 20 \vdash \xi 101} \star}{\vdash \xi 2, \xi 100} \quad \frac{\xi 10 \vdash \xi 2}{\vdash \xi 1, \xi 2}}{\xi \vdash} & \mathfrak{G}'' &= \frac{\frac{\frac{\vdash \xi 100, \xi 200}{\xi 10 \vdash \xi 200} \star \quad \frac{\vdash \xi 101, \xi 201}{\xi 10 \vdash \xi 201} \star}{\vdash \xi 1, \xi 200} \quad \frac{\xi 20 \vdash \xi 1}{\vdash \xi 1, \xi 2}}{\xi \vdash} \end{aligned}$$

The set  $V_{\mathbf{G}}$  is defined as the positive-prefix closed and daimon-prefix closed set given by the four following paths:

$$\begin{aligned}
& (+, \xi, \{1, 2\})(-, \xi 1, 0)(+, \xi 10, 0)(-, \xi 2, 0)(+, \xi 20, 0), \\
& (+, \xi, \{1, 2\})(-, \xi 2, 0)(+, \xi 20, 0)(-, \xi 1, 0)(+, \xi 10, 0), \\
& (+, \xi, \{1, 2\})(-, \xi 1, 0)(+, \xi 10, 1)(-, \xi 2, 0)(+, \xi 20, 1), \\
& (+, \xi, \{1, 2\})(-, \xi 2, 0)(+, \xi 20, 1)(-, \xi 1, 0)(+, \xi 10, 1).
\end{aligned}$$

Remark that the following designs do not belong to  $\mathbf{G}$ :

$$\begin{aligned}
\mathfrak{G}_1 &= \frac{\frac{\xi 100 \vdash}{\vdash \xi 10} \quad \frac{\xi 201 \vdash}{\vdash \xi 20}}{\xi 1 \vdash \quad \xi 2 \vdash} \vdash \xi & \mathfrak{G}_2 &= \frac{\frac{\xi 101 \vdash}{\vdash \xi 10} \quad \frac{\xi 200 \vdash}{\vdash \xi 20}}{\xi 1 \vdash \quad \xi 2 \vdash} \vdash \xi
\end{aligned}$$

The behaviour  $\mathbf{G}$  satisfies the following properties:  $C_{\mathbf{G}} \subset V_{\mathbf{G}}$ ,  $S_{\mathbf{G}^\perp} = C_{\mathbf{G}^\perp} \subset V_{\mathbf{G}^\perp}$  but  $S_{\mathbf{G}} \not\subset V_{\mathbf{G}}$ . Hence  $\mathbf{G}$  is not regular.

- Finally, let us consider the behaviour  $\mathbf{H} = \{\mathfrak{E}, \mathfrak{H}\}^{\perp\perp}$  where designs  $\mathfrak{E}$  and  $\mathfrak{H}$  are given below. The dual behaviour is  $\mathbf{H}^\perp = \{\mathfrak{H}'\}^{\perp\perp}$  where the design  $\mathfrak{H}'$  is given below on the second line.

$$\begin{aligned}
\mathfrak{E} &= \frac{\frac{\xi 100 \vdash}{\vdash \xi 10} \quad \frac{\xi 200 \vdash}{\vdash \xi 20}}{\xi 1 \vdash \quad \xi 2 \vdash} \vdash \xi & \text{and} & \quad \mathfrak{H} = \frac{\frac{\xi 101 \vdash}{\vdash \xi 10} \quad \frac{\xi 211 \vdash}{\vdash \xi 21}}{\xi 1 \vdash \quad \xi 2 \vdash} \vdash \xi
\end{aligned}$$

$$\begin{aligned}
& \frac{\frac{\frac{\vdash \xi 100, \xi 200}{\xi 20 \vdash \xi 100} \quad \frac{\vdash \xi 101, \xi 211}{\xi 21 \vdash \xi 101}}{\vdash \xi 2, \xi 100} \quad \frac{\vdash \xi 2, \xi 101}}{\xi 10 \vdash \xi 2} \\
& \frac{\vdash \xi 1, \xi 2}{\xi \vdash} \\
\mathfrak{H}' &=
\end{aligned}$$

The set  $V_{\mathbf{H}}$  is defined as the daimon-prefix closed set given by two paths:

$$\begin{aligned}
& (+, \xi, \{1, 2\})(-, \xi 1, 0)(+, \xi 10, 0)(-, \xi 2, 0)(+, \xi 20, 0), \\
& (+, \xi, \{1, 2\})(-, \xi 1, 0)(+, \xi 10, 1)(-, \xi 2, 1)(+, \xi 21, 1).
\end{aligned}$$

We have that  $C_{\mathbf{H}} \not\subset V_{\mathbf{H}}$ : the behaviour  $\mathbf{H}$  is not regular.

We finish this section with a proposition that summarizes previous results: we relate regularity and logical operations on behaviours.

**Proposition 4.11**  $\mathbf{A}$  is a regular positive connected behaviour iff

- either  $\mathbf{A} = \mathbf{1}$ ,
- or  $\mathbf{A} = \downarrow \mathbf{B}$  and  $\mathbf{B}$  is a regular negative behaviour,
- or  $\mathbf{A} = \mathbf{D} \otimes \mathbf{E}$  where  $\mathbf{D}$  and  $\mathbf{E}$  are regular alien positive behaviours.

*Proof* The ‘if’ part follows from proposition 4.7. Let us suppose that  $\mathbf{A}$  is a regular positive connected behaviour. Recall that a behaviour is connected if its directory is a singleton  $I$ , i.e. the first action of its designs is  $(+, \xi, I)$ .

- If  $I = \emptyset$ ,  $\mathbf{A} = \mathbf{1}$ .
- If  $I = \{i\}$ , then designs in  $\mathbf{A}$  are of the form  $(+, \xi, i)\mathfrak{D}$ . Normalization being deterministic and  $\mathbf{A}$  closed, the set  $\mathbf{B}$  of such designs  $\mathfrak{D}$  is a behaviour. Hence  $\mathbf{A} = \downarrow \mathbf{B}$ . We conclude with proposition 4.7.

- Otherwise, let  $K \subset I$  such that  $K$  and  $I - K$  are non-empty. We note the following point: we can always consider that a design in  $\mathbf{A}$  is a tensor of a design with first action  $(+, \xi, K)$  and a design with first action  $(+, \xi, I - K)$ . We consider the set  $\mathbf{D}$  (resp.  $\mathbf{E}$ ) of such designs of first action  $(+, \xi, K)$  (resp.  $(+, \xi, I - K)$ ). Let  $\mathfrak{D}_1 \in \mathbf{D}$  and  $\mathfrak{E}_2 \in \mathbf{E}$ , the two distinct from  $\mathfrak{D}\mathfrak{a}\mathfrak{i}$ . Hence there exist designs  $\mathfrak{D}_2 \in \mathbf{D}$  and  $\mathfrak{E}_1 \in \mathbf{E}$  such that  $\mathfrak{D}_1 \otimes \mathfrak{E}_1 \in \mathbf{A}$  and  $\mathfrak{D}_2 \otimes \mathfrak{E}_2 \in \mathbf{A}$ . Let  $\mathfrak{F} \in \mathbf{A}^\perp$ . Suppose that  $\mathfrak{F} \not\leq \mathfrak{D}_1 \otimes \mathfrak{E}_2$ . Let us consider the path  $p$  obtained during the normalisation of  $\mathfrak{D}_1 \otimes \mathfrak{E}_2$  with  $\mathfrak{F}$  before divergence.
  - The path  $p$  cannot be empty as the first action of  $\mathfrak{D}_1 \otimes \mathfrak{E}_2$  is  $(+, \xi, I)$  and the first action of  $\mathfrak{F}$  is  $(-, \xi, I)$ . Hence we can write  $p = (+, \xi, I)p'$ .
  - If the last action of  $p$  is a negative action  $\kappa^-$ , w.l.o.g. we can suppose that  $\lceil (+, \xi, K)p' \rceil \in \mathfrak{D}_1$  and there exists a (unique) positive action  $\kappa^+$  that extends  $\lceil (+, \xi, K)p' \rceil$  in  $\mathfrak{D}_1$ . Note that  $\lceil (+, \xi, I)p'\kappa^+ \rceil$  is a set of chronicles in  $\mathfrak{D}_1 \otimes \mathfrak{E}_1 \cup \mathfrak{D}_2 \otimes \mathfrak{E}_2$ . Hence, as  $\mathbf{A}$  is regular, positive-ended chronicles in  $\lceil p\kappa^+ \rceil$  are in  $V_{\mathbf{A}}$ .
    - If  $\kappa^+$  is initial in  $\mathfrak{D}_1 \otimes \mathfrak{E}_2$ , then one can define the design  $\mathfrak{F}' = \mathfrak{F} \cup \{\overline{\kappa^+}\}$  and we have that  $\widetilde{p\kappa^+} \in \mathcal{P}_{\mathfrak{F}'}$ , in particular  $\widetilde{p\kappa^+}$  is a path. Furthermore, there exist paths  $q \in \mathcal{P}_{\mathfrak{D}_1 \otimes \mathfrak{E}_1}$  and  $r \in \mathcal{P}_{\mathfrak{D}_2 \otimes \mathfrak{E}_2}$  such that  $p\kappa^+ \in q \sqcup r$ . Thus, as  $\mathbf{A}$  is regular,  $p\kappa^+ \in V_{\mathbf{A}}$ . Hence contradiction as  $\widetilde{p} \in \mathcal{P}_{\mathfrak{F}}$  and  $\widetilde{p\kappa^+} \notin \mathcal{P}_{\mathfrak{F}}$ .
    - Otherwise there exists a negative action  $\kappa_0^-$  that justifies  $\kappa^+$ . There exist paths  $p_1$  and  $p_2$  such that  $p = p_1\kappa_0^-p_2\kappa^+$ . Hence  $\lceil p_1\kappa_0^- \rceil \in \mathfrak{F}$ . Let us define the design  $\mathfrak{F}' = \mathfrak{F} \cup \lceil p_1\kappa_0^-\kappa^+ \rceil$  then  $\widetilde{p\kappa^+} \in \mathcal{P}_{\mathfrak{F}'}$ . With the same reasoning as in the previous item, this yields  $p\kappa^+ \in V_{\mathbf{A}}$ , hence a contradiction.
  - If the last action of  $p$  is a positive action. This positive action should be distinct from  $\star$  otherwise the normalisation succeeds. We have that  $\overline{p} \in \mathcal{P}_{\mathfrak{F}}$ , hence there exists a (unique) positive action  $\kappa^+$  that extends the chronicle  $\lceil \overline{p} \rceil$  in  $\mathfrak{F}$ . Note that  $\lceil \overline{p\kappa^+} \rceil$  is a set of chronicles in  $\mathfrak{F}$ . Hence, as  $\mathbf{A}$  is regular, positive-ended chronicles in  $\lceil \overline{p\kappa^+} \rceil$  are in  $V_{\mathbf{A}^\perp}$ .  $\kappa^+$  cannot be initial in  $\mathfrak{F}$  as  $\mathfrak{F}$  is a negative behaviour (of regular base). Hence there exists a negative action  $\kappa_0^+$  that justifies  $\kappa^+$ . There exist paths  $\overline{p_1}$  and  $\overline{p_2}$  such that  $\overline{p} = \overline{p_1\kappa_0^+p_2\kappa^+}$ . W.l.o.g. we can suppose that  $\lceil p_1\kappa_0^+ \rceil \in \mathfrak{D}_1$ . Let us define the design  $\mathfrak{D}'_1 = \mathfrak{D}_1 \cup \lceil p_1\kappa_0^+\kappa^+ \rceil$  then  $\overline{p\kappa^+} \in \mathcal{P}_{\mathfrak{D}'_1}$ . Thus, as  $\mathbf{A}$  is regular,  $\overline{p\kappa^+} \in V_{\mathbf{A}}$ , hence also  $\lceil p\kappa^+ \rceil$ . Contradiction with the fact that  $\lceil p\kappa^+ \rceil \notin \mathfrak{D}_1$ .

Thus  $\mathfrak{D}_1 \otimes \mathfrak{E}_2 \perp \mathfrak{F}$ , hence  $\mathfrak{D}_1 \otimes \mathfrak{E}_2 \in \mathbf{A}$ . So we have  $\mathbf{D} \otimes \mathbf{E} \subset \mathbf{A}$ . Hence  $\mathbf{D}$  and  $\mathbf{E}$  are behaviours and  $\mathbf{D} \otimes \mathbf{E} = \mathbf{A}$ .

Finally, the fact that  $\mathbf{D}$  and  $\mathbf{E}$  are regular follows from proposition 4.7.

## 5 Essentially Finite Behaviours, Uniformly Bounded Behaviours

The aim of this section is to characterize behaviours that may be *finitely* decomposed by means of logical operations on behaviours (tensor, plus, shift and their duals). Finiteness and boundedness characterize among the behaviours the ones on which additive and multiplicative decompositions yield atomic behaviours. These latter are the behaviours associated with the linear constants 1, 0,  $\perp$  and  $\top$ . In other words, finiteness or boundedness guarantee that such additive and multiplicative decompositions terminate: incarnated designs in such behaviours are trees of finite height.

**Definition 5.1 (Size of a design, of a net)** *The size of a design  $\mathcal{D}$ , noted  $\#\mathcal{D}$ , is the number of proper actions of this design, infinite if there is an infinite number of proper actions. The size of a net of designs is the sum of sizes of its designs.*

**Definition 5.2 (Finite Design)** *A design  $\mathcal{D}$  is finite if its size is finite, i.e.,  $\#\mathcal{D} < +\infty$ .*

It is easy to prove that a design is finite iff its number of chronicles is finite iff it has a finite number of finite slices iff there is a finite number of occurrences of foci.

**Definition 5.3 (Essential finiteness / Uniform boundedness)** *A behaviour  $\mathbf{G}$  is essentially finite if  $\sum_{\mathcal{D} \in |\mathbf{G}|} \#\mathcal{D} < +\infty$ .*

*A behaviour  $\mathbf{G}$  is uniformly bounded if there exists  $N$  such that for all design  $\mathcal{D} \in |\mathbf{G}|$ , for all slice  $\mathcal{S}$  of  $\mathcal{D}$ ,  $\#\mathcal{S} < N$ .*

In other words, a behaviour is essentially finite iff it has a finite number of material designs, and these designs are finite. A uniformly bounded behaviour may have an infinite number of material designs and these latters may have an infinite number of slices but these latters are finite. Then the material designs in a uniformly bounded behaviour have finite height. Note that an essentially finite behaviour is uniformly bounded. We recall in the following lemma properties concerning incarnation of behaviours.

**Lemma 5.4** [5, 10] *Let  $\mathbf{G}$  and  $\mathbf{H}$  be two positive behaviours,*

*– If  $\mathbf{G}$  is of base  $\vdash \xi.i, \Lambda$  then  $|\uparrow\mathbf{G}| = (-, \xi, \{i\}) \cdot |\mathbf{G}|$*

*– If  $\mathbf{G}$  and  $\mathbf{H}$  are disjoint then  $|\mathbf{G} \oplus \mathbf{H}| = |\mathbf{G}| \cup |\mathbf{H}|$*

*– If  $\mathbf{G}$  and  $\mathbf{H}$  are alien then  $|\mathbf{G} \otimes \mathbf{H}| = |\mathbf{G}| \odot |\mathbf{H}|$*

*where  $\mathbf{G} \odot \mathbf{H} = \{\mathcal{D} \otimes \mathcal{E} ; \mathcal{D} \in \mathbf{G}, \mathcal{E} \in \mathbf{H}\}$*

**Proposition 5.5 (Ess. finiteness: stability properties)** *Let  $\mathbf{P}$  and  $\mathbf{Q}$  be positive behaviours,*

*–  $\mathbf{P}$  is ess. finite iff  $\mathbf{P}^\perp$  is ess. finite,*

*–  $\mathbf{P}$  is ess. finite iff  $\uparrow\mathbf{P}$  is ess. finite,*

*– let  $\mathbf{P}$  and  $\mathbf{Q}$  be disjoint,  $\mathbf{P}$  and  $\mathbf{Q}$  are ess. finite iff  $\mathbf{P} \oplus \mathbf{Q}$  is ess. finite,*

*– let  $\mathbf{P}$  and  $\mathbf{Q}$  be alien,  $\mathbf{P}$  and  $\mathbf{Q}$  are ess. finite iff  $\mathbf{P} \otimes \mathbf{Q}$  is ess. finite.*

*Proof* Let  $\mathbf{P}$  be an ess. finite behaviour. The incarnation  $|\mathbf{P}^\perp|$  of  $\mathbf{P}^\perp$  is the set of nets  $\mathfrak{R}$  such that [9]:

$$\mathfrak{R} = \bigcup_{\mathcal{D} \in \mathbf{P}} {}^\top\langle \mathfrak{R} \leftarrow \mathcal{D} \rangle^\top = \bigcup_{\mathcal{D} \in |\mathbf{P}|} {}^\top\langle \mathfrak{R} \leftarrow \mathcal{D} \rangle^\top$$

Note that for all  $\mathcal{D} \in \mathbf{P}$ , the normalization path  ${}^\top\langle \mathfrak{R} \leftarrow \mathcal{D} \rangle^\top$  only consists of the daimon or proper actions of  $\mathcal{D}$  where the polarity has been changed. Hence, when  $\mathcal{D} \in |\mathbf{P}|$ , the size of  $\mathcal{D}$  is finite, so is the size of  ${}^\top\langle \mathfrak{R} \leftarrow \mathcal{D} \rangle^\top$ . Finally, as the number of material designs in  $\mathbf{P}$  is finite, the size of  $\mathfrak{R}$  is finite.

Each such anti-design  $\mathfrak{R}$  has proper actions taken from the same finite set of proper actions: the set of dual proper actions that appear in material designs of  $\mathbf{P}$ . Hence such each anti-design  $\mathfrak{R}$  has a finite number of slices. In each of its slices, proper actions are distinct. Thus there is a finite number of slices that can be ‘built’. As a consequence, the number of material nets in  $\mathbf{P}^\perp$  is finite:  $\mathbf{P}^\perp$  is ess. finite.

The other items are immediate consequences of lemma 5.4.



**Proposition 5.6 (Unif. boundedness: stability)** *Let  $\mathbf{P}$  and  $\mathbf{Q}$  be positive behaviours,*

- $\mathbf{P}$  is unif. bounded iff  $\mathbf{P}^\perp$  is unif. bounded,*
- $\mathbf{P}$  is unif. bounded iff  $\uparrow \mathbf{P}$  is unif. bounded,*
- let  $\mathbf{P}$  and  $\mathbf{Q}$  be disjoint,  $\mathbf{P}$  and  $\mathbf{Q}$  are unif. bounded iff  $\mathbf{G} \oplus \mathbf{H}$  is unif. bounded,*
- let  $\mathbf{P}$  and  $\mathbf{Q}$  be alien,  $\mathbf{P}$  and  $\mathbf{Q}$  are unif. bounded iff  $\mathbf{G} \otimes \mathbf{H}$  is unif. bounded.*

*Proof* Let  $\mathbf{P}$  be a behaviour unif. bounded by  $N$ . Let  $\mathfrak{R} \in |\mathbf{P}^\perp|$  and  $\mathfrak{D} \in |\mathbf{P}|$ . Note that the normalization path  $\langle \mathfrak{D} \leftarrow \mathfrak{R} \rangle$  is in a unique slice of  $\mathfrak{D}$ . Hence  ${}^\top \langle \mathfrak{R} \leftarrow \mathfrak{D} \rangle^\top$  has size less than  $N$ . Therefore slices of  $\mathfrak{R} = \bigcup_{\mathfrak{D} \in |\mathbf{P}|} {}^\top \langle \mathfrak{R} \leftarrow \mathfrak{D} \rangle^\top$  have sizes less than  $N$ , thus  $\mathbf{P}^\perp$  is unif. bounded by  $N$ .

The remainder of the proof is a consequence of lemma 5.4.

## 6 Behaviours associated with Logical Formulas

In all this section, we consider that the base of a behaviour is either  $\vdash \xi$  or  $\xi \vdash$ . We consider two families of behaviours  $\mathcal{C}_f = \mathcal{C}_f^+ \cup \mathcal{C}_f^-$  and  $\mathcal{C}_\infty = \mathcal{C}_\infty^+ \cup \mathcal{C}_\infty^-$  defined inductively in the following way:

$$\begin{aligned} \mathcal{C}_f^+ &= \mathbf{0} | \mathbf{1} | \bigoplus_{n \in [1, \infty[} (\bigotimes_{q \in [1, \infty[} \downarrow \mathcal{C}_f^-) & \mathcal{C}_f^- &= \top | \perp | \bigotimes_{n \in [1, \infty[} (\mathfrak{Y}_{q \in [1, \infty[} \uparrow \mathcal{C}_f^+) \\ \mathcal{C}_\infty^+ &= \mathbf{0} | \mathbf{1} | \bigoplus_{n \in [1, \infty[} (\bigotimes_{q \in [1, \infty[} \downarrow \mathcal{C}_\infty^-) & \mathcal{C}_\infty^- &= \top | \perp | \bigotimes_{n \in [1, \infty[} (\mathfrak{Y}_{q \in [1, \infty[} \uparrow \mathcal{C}_\infty^+) \end{aligned}$$

The aim of this section is to characterize elements of  $\mathcal{C}_\infty$  and  $\mathcal{C}_f$ . More precisely, we state in theorem 6.1 that behaviours of  $\mathcal{C}_\infty$  (resp.  $\mathcal{C}_f$ ) are exactly the regular and uniformly bounded (resp. ess. finite) behaviours. It is immediate that  $\mathcal{C}_f \subset \mathcal{C}_\infty$ :  $\mathcal{C}_\infty$  differs from  $\mathcal{C}_f$  by allowing infinite additive structures. Note furthermore that the set  $\mathcal{C}_f$  is sound and complete with respect to the set of polarized multiplicative-additive formulas: this result is a straightforward corollary of Girard's result for **MALL**<sub>2</sub>.

### Theorem 6.1

$\mathbf{A} \in \mathcal{C}_\infty$  iff  $\mathbf{A}$  is a regular uniformly bounded behaviour.  
 $\mathbf{A} \in \mathcal{C}_f$  iff  $\mathbf{A}$  is a regular essentially finite behaviour.

*Proof* Let us first consider that  $\mathbf{A} \in \mathcal{C}_\infty$ . The proof that  $\mathbf{A}$  is uniformly bounded and regular is done by induction on the structure of  $\mathbf{A}$ :

- The property is immediately satisfied for the constants  $\mathbf{0}$ ,  $\mathbf{1}$ ,  $\top$ ,  $\perp$ .
- The induction follows from lemmas 4.4 and 4.7 for regularity, and proposition 5.6 for uniform boundedness.

Let us suppose that  $\mathbf{A}$  is regular and uniformly bounded by  $N$ . We prove that  $\mathbf{A} \in \mathcal{C}_\infty$  by induction on  $N$ :

- The property is immediately satisfied for  $N = 0, 1$  or  $2$  (corresponding to constants  $\mathbf{0}$ ,  $\mathbf{1}$ ,  $\top$ ,  $\perp$ ).
- Suppose the property true until  $N$  and let  $\mathbf{A}$  be a regular behaviour uniformly bounded by  $N + 1$ . Suppose  $\mathbf{A}$  is positive. Following the additive decomposition theorem ([10], Th. 12), there exists a family of connected behaviours  $(\mathbf{A}_k)_{k \in K}$  such that  $\mathbf{A} = \bigoplus_{k \in K} \mathbf{A}_k$ . Note that for each  $k \in K$ ,  $\mathbf{A}_k$  is uniformly bounded by  $N + 1$ . By lemma 4.11, we can split  $K$  into three sets  $K_1, K_2, K_3$  such that for each  $k \in K_1$ ,  $\mathbf{A}_k = \mathbf{1}$ , for each  $k \in K_2$ ,  $\mathbf{A}_k = \downarrow \mathbf{A}'_k$ , where  $\mathbf{A}'_k$  is a regular negative

behaviour, and for each  $k \in K_3$ ,  $\mathbf{A}_k = \bigotimes_{l_k \in L_k} \mathbf{A}_{kl_k}$  where  $L_k$  is the ramification of the first action, and  $\mathbf{A}_{kl_k}$  is a regular behaviour. Furthermore, the ramification of each  $\mathbf{A}_{kl_k}$  is a singleton: it is a shift of a negative behaviour  $\mathbf{A}'_{kl_k}$ . Finally we notice that each  $\mathbf{A}'_k$  and  $\mathbf{A}_{kl_k}$  is uniformly bounded by  $N$ . Hence the induction applies on these behaviours. We end noticing that  $\mathbf{A} = \bigoplus_{k \in K_1} \mathbf{1} \oplus \bigoplus_{k \in K_2} \downarrow \mathbf{A}'_k \oplus \bigoplus_{k \in K_3} \bigotimes_{l_k \in L_k} \downarrow \mathbf{A}_{kl_k}$ . When  $\mathbf{A}$  is a negative behaviour, it suffices to consider its dual  $\mathbf{A}^\perp$ .

The other property is proved similarly using proposition 5.5.

## 7 Conclusion

At first sight, Ludics objects seem to be easy to study: designs are nothing else but abstraction of proofs or counter-proofs of multiplicative-additive Linear Logic (MALL). However, this is not the case when one tries to identify among behaviours, *i.e.*, closures of sets of designs, those that are interpretation of MALL formulas. This paper is a first step toward a full algebraic study of behaviours. First, we make explicit the equivalence between the two presentations of a design, as set of paths versus set of chronicles. We give a few properties concerning orthogonality in terms of path travelling, introducing visitable paths, *i.e.*, paths that are visited by orthogonality. Finally, our main result is a characterization of  $\mathcal{C}_\infty$  behaviours that correspond to formulas built from the linear constants by means of infinitely additive connectives and multiplicative connectives. In particular, we show that such behaviours should satisfy a notion of *regularity*. Regularity in turn may be defined as a global property: roughly speaking, reversible paths built from actions in the incarnation should be visitable. Regularity may also be defined in a constructive way: visitable paths are exactly chronicles of the incarnation or reversible shuffles of such paths. Such a study should help us understanding the structure of MALL proofs.

Let us remark finally that properties of Ludics that serve for proving that Ludics is a fully abstract model of MALL<sup>4</sup>, are satisfied for the entire Ludics and not only for behaviours interpreting MALL formulas: interaction between objects, that is cut-elimination, is at the heart of Ludics, thus it allows to consider the Ludics framework as a semantics for computation beyond what is given with MALL: a behaviour may model a type and (open) interaction between behaviours corresponds to composition of types. For future work, we plan to extend our analysis to the whole set of behaviours, defining a grammar for it in such a way that connectives of the grammar may be computationnally (or logically) interpreted.

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<sup>4</sup>More precisely of a slightly modified polarized second-order MALL.

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